Introduction to Algebra

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Abstract

An introduction to algebra shall enable the approach also to those, who somewhen have lost the trail in school, or who have joy in direct calculation ways. The threefold proof, which is embodied in at least three divers cultures, thereby enables to keep the overview, and to find own mistakes as quickly as possible.

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1 Of all Good Things are Three

1.1 German Proverb

An old German proverb reads:

> *Aller guten Dinge sind drei—Of all good things are three.*

This means, if something is correct, then exist at least three independent approaches to it, like each mountain summit has got at least three ridges. The aforementioned proverb has been handed down in German without date and belongs like the German language and the local landscape names to the oldest, cultural reports in Germany.
But now, also the German proverb is just a single source, thus it's worth looking for further sources with the same content.

1.2 Chinese Character for Quality

Already early, the Chinese people has handed down with its language very many characters, which consist as symbol collection of more easier symbols and sometimes explain contexts, which were known at that time. So, the traditional, Chinese character 品[1] for *range, class, personality*[2] consists of 3 characters 口[3] for *mouth, opening, persons*[4]:

> 品是三口[5].

This sentence means:

> *Quality is based on three mouths.*

These characters and their corresponding meaning are used without change also in Japan[6] and in the modern China[7].

1.3 Argumentation due to Moses

In Israel since *Moses*[8] there is the principle, that an argumentation being on trial is valid by the combination of the mouth of two or three witnesses:

> *One witness shall not rise up against a man*
> *for any iniquity, or for any sin,*
> *in any sin that he sinneth:*
> *at the mouth of two witnesses, or at the mouth of three witnesses,*
> *shall the matter be established.*

[1] pronunciation: *pin3*
[2] [1924Rüd], number 4321, page 425
[3] pronunciation: *kou3*
[4] [1924Rüd], number 3243, page 334
[5] pronunciation: *pin3 shi4 san1 kou3*
[6] [1994Had], number 230 and 54, page 99 and 74
[7] [1993XYCGZZDYCKN], page 620 and 471
[8] [1994AV], Deuteronomy 19:15

This principle is found all over the whole Bible. So it is mentioned as base of God's *Trinity*[9], where only *witnesses* of the argumentation are mentioned, and not 3 persons[10]:

> *7. For there are three that bear record in heaven,*
> *the Father, the Word, and the Holy Ghost: and these three are one.*
> *8. And there are three that bear witness in earth,*
> *the Spirit, and the water, and the blood: and these three agree in one.*

The lastest since the Enlightment, the argumentation due to Moses is scorned in Europe, because in contrast to God, each human is a single witness only and needs the confirmation by other witnesses. Therefore humans, who thing themselves to be important, argue quite differently, and unfortunately they wait only sometimes, until their results are confirmed independently by the second and the third side. Also such variants are not forbidden, but often enough they lead into the self–elected fallacy.
At scientific conferences usually the researchers tell their current position to the collegues for discussion. During this can occur, that they are confirmed by collegues with similar results, or however, that the others localize the fundamental errors. In no science there is a tradition to vote democratically for correctness. Rather the principle is valid:

> *Whosoever does not bear to be smiled at by the collegues,*
> *the same should not do research.*

So, Nikolai Kopernik[11] held his life's work in his hands not before being on his death–bed.

1.4 Conclusion for the Assessment of a Calculation

Thus now 3 independent sources are proven, which use the combination of 3 witnesses for the correctness of a statement. Being on courts, the argumentation may be valid less strictly as in research, instruction, and industrial quality management, therefore on courts Moses prescribes the coincidence of at least two witnesses.
Here, the insight, won by this, is used to exercise and rule enough variants to solve an algebraic task. In all fields, an expert is recognized by knowing about alternatives. These alternatives are presented by the principle of the 3 solution ways and enable to deepen the lesson by further calculation ways.
On the question, whether such thing would be possible in general, Jesus Christ gives the following information[12]:

> *If thou canst believe, all things are possible to him that believeth.*

Believing mainly means in the Bible: *to be told.* Therefore, the search for the three solution ways is worth to be done. How long the search will last, stays to be exciting.

[9] The notion *triple unity* occurs quite late in theology and claims the mathematical nonsense: $3 = 1$ ([2007Ryr], chapter 8.II.c, page 82). Already *Isaac Newton* had problems with this non–biblical subtlety ([2009GB], chapter 10, page 114), which is based on the scholastic dogmatizing of *Aristoteles* during the Middle Ages.
[10] [1994AV], 1 John 5:7–8
[11] also known as *Copernicus*, [1953VEB], entry *Kopernik(us)*, page 542
[12] see [1994AV], St. Mark 9:23

2 Algebraic Equations

2.1 Algebraic Equation of 1^{st} Degree

2.1.1 The Equation

An *algebraic equation* of first degree is given by the following equation:

$$a\,x + b = 0 \tag{1}$$

Here, a and b are independent of the yet unknown solution x.

2.1.2 1^{st} Solution Way

Algebra is living by the remaining of the equality sign of an equation, if on each of both sides the same calculation operation is done. Here, this yields the following calculation steps:

$$\begin{aligned} a\,x &= -b \qquad \Leftrightarrow \\ x &= -\frac{b}{a}\,. \end{aligned} \tag{2}$$

2.1.3 2^{nd} Solution Way

Another solution way results by *substitution*[13]:

$$\begin{aligned} x &\rightarrow y - \frac{b}{a} \qquad \Rightarrow \\ a\left(y - \frac{b}{a}\right) + b &= a\,y - b + b \,=\, a\,y \,=\, 0 \qquad \Leftrightarrow \\ y &= 0 \qquad \Rightarrow \\ x &= 0 - \frac{b}{a} = -\frac{b}{a}\,. \end{aligned} \tag{3}$$

2.1.4 3^{rd} Solution Way

The third solution way divides the equation by x, solves to $\frac{1}{x}$, and then builds the reciprocal:

$$\begin{aligned} a + \frac{b}{x} &= 0 \qquad \Leftrightarrow \\ \frac{b}{x} &= -a \qquad \Leftrightarrow \\ \frac{1}{x} &= -\frac{a}{b} \qquad \Leftrightarrow \\ x &= -\frac{b}{a}\,. \end{aligned} \tag{4}$$

[13] replacement

2.1.5 Checking Calculation

The 3 solution ways lead to the same solution (2), (3) und (4). Whether this solution is valid, always only shows the *checking calculation* in the starting equation, here equation (1):

$$\begin{aligned} a\left(-\frac{b}{a}\right)+b &= -b+b = 0 \qquad \Leftrightarrow \\ 0 &= 0 \end{aligned} \tag{5}$$

Zero is zero for each choice of all parameters. Thus the equation is solved.
This needed to be shown[14].

2.1.6 Ishmael's Algebra

Ishmael[15], the son of Abraham, was sent[16] with his mother into the desert[17]. To survive there, he needed to handle the food for the journey with care. Subsequently resulted the *arabic numbers*, which can reliably deal with arbitrary, huge quantities. As well he searched for *the number*[18] of camels, which are needed to carry a scheduled commodity amount during an intended journey duration through a desert. In 1202, the corresponding calculation art was translated into Latin and expanded by *Leonardo da Pisa*, called *Fibonacci*[19], and since then it is written *algebra* and is pronounced still the same as in Arabic. The solution of the historic task with world–wide importance begins with 2 equations for 2 unknown variables:

$$\text{total\_burden} = \text{load\_capacity} \cdot \text{camel\_number}\,. \tag{6}$$

$$\text{total\_burden} = \text{ware\_weight} + \text{victual\_need} \cdot \text{duration} \cdot \text{camel\_number}\,. \tag{7}$$

At these equations, the total burden and the camel number are unknown. Now both left hand sides are the same, thus the right hand sides of both equations are equal:

$$\begin{aligned} &\text{load\_capacity} \cdot \text{camel\_number} \\ =\ &\text{ware\_weight} + \text{victual\_need} \cdot \text{duration} \cdot \text{camel\_number}\,. \end{aligned} \tag{8}$$

Now, an equality sign stays valid, if on each side of an equation is done the same. This yields the following rearrangement of equation (8), that yet contains an unknown camel number only, which is sensible until a maximum duration:

$$\begin{aligned} \text{ware\_weight} &= (\text{load\_capacity} - \text{victual\_need} \cdot \text{duration}) \cdot \text{camel\_number} \qquad \Leftrightarrow \\ \text{camel\_number} &= \frac{\text{ware\_weight}}{\text{load\_capacity} - \text{victual\_need} \cdot \text{duration}}\,. \end{aligned} \tag{9}$$

The found solution fulfills the equation (8) with the result $0 = 0$ and leads from the equations (6) and (7) in each case to the same total burden:

$$\text{total\_burden} = \frac{\text{load\_capacity} \cdot \text{ware\_weight}}{\text{load\_capacity} - \text{victual\_need} \cdot \text{duration}}\,. \tag{10}$$

[14] Latin version: *quod erat demonstrandum.*
[15] born about 2085 before Christ, died about 1948 before Christ
[16] [1994AV], Genesis 21:10–21
[17] Hebrew: *arab*
[18] Arabic: *al–Djabr* means about: *The compellingly* needed calculation way.
[19] [1959Mesch], section I 1., page 9–10

2.1.7 Checking Calculations

In a desert, there is only one trial to check the correctness of a calculation. Therefore, no fallacies are to be applied here. As alternative calculation way, here the reciprocals of the equations present themselves:

$$\frac{1}{\text{total\_burden}} = \frac{1}{\text{load\_capacity} \cdot \text{camel\_number}}\,. \tag{11}$$
$$\frac{1}{\text{total\_burden}} = \frac{1}{\text{ware\_weight} + \text{victual\_need} \cdot \text{duration} \cdot \text{camel\_number}}\,. \tag{12}$$

Equating of (11) and (12) leads to the following result:

$$1 = \frac{\text{ware\_weight} + \text{victual\_need} \cdot \text{duration} \cdot \text{camel\_number}}{\text{load\_capacity} \cdot \text{camel\_number}} \quad \Leftrightarrow$$
$$\frac{\text{ware\_weight}}{\text{camel\_number}} = \text{load\_capacity} - \text{victual\_need} \cdot \text{duration}\,. \tag{13}$$

The result (13) can be solved to the result (9), which completes an independent calculation way.
A third calculation way results by dividing each of the equations (6) and (7) by the camel number, and then equating them:

$$\text{load\_capacity} = \frac{\text{ware\_weight}}{\text{camel\_number}} + \text{victual\_need} \cdot \text{duration} \quad \Leftrightarrow$$
$$\frac{\text{ware\_weight}}{\text{camel\_number}} = \text{load\_capacity} - \text{victual\_need} \cdot \text{duration}\,. \tag{14}$$

The result (14) is identical to (13) and leads in each case to the solution (9). Also for this transition, several variants are possible, either directly, or by solving to the reciprocal of the camel number and subsequent reciprocal at both sides of the equation.

2.1.8 Importance

This is the beginning of algebra and the trade caravans through the deserts of this earth. Due to the report of the Holy Scriptures, the career of Ishmael is connected to a divine blessing, which Abraham asked for his son[20]:

> *And as for Ishmael, I have heard thee:*
> *Behold, I have blessed him,*
> *and will make him fruitful,*
> *and will multiply him exeedingly;*
> *twelve princes shall he beget,*
> *and I will make him a great nation.*

Therefore, whosoever is sent by others into the desert, the same especially there is able to experience the blessing of the Most High[21].

[20] [1994AV], Genesis 17:20
[21] [1994AV], Psalm 84:5–7

2.2 Algebraic Equation of 2^{nd} Degree

2.2.1 The Equation

An *algebraic equation* of 2^{nd} degree is given by the following equation:

$$a\,x^2 + b\,x + c = 0\,. \tag{15}$$

Here, a, b, and c are independent of the yet unknown solution x.

2.2.2 1^{st} Solution Way

Here, by the 4 basic arithmetic operations no solution is found, rather a *square root* must be calculated, which leads from the fractions to the real valued and even complex valued numbers. For this at first a *reduced* equation is built by skilful substitution, which follows from the *binomial theorem*:

$$\begin{aligned}
x &\;\rightarrow\; y - \frac{b}{2\,a} && \Rightarrow \\
a\left(y^2 - \frac{b}{a}\,y + \frac{b^2}{4\,a^2}\right) + b\left(y - \frac{b}{2\,a}\right) + c &= 0 && \Leftrightarrow \\
a\,y^2 - b\,y + b\,y + \frac{b^2}{4\,a} - \frac{b^2}{2\,a} + c &= 0 && \Leftrightarrow \\
y^2 &= \frac{b^2}{4\,a^2} - \frac{c}{a} && \Leftrightarrow \\
y &= \pm\sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}} && \Rightarrow \\
x &= -\frac{b}{2\,a} \pm \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\,.
\end{aligned} \tag{16}$$

By correct application, algebra is always applicable to all equations with complex coefficients and leads via the *square root* to solutions of the *complex numbers*, if beneath the root in equation (16) is found a negative or complex number.

2.2.3 2^{nd} Solution Way

Here, the equation is divided first by x^2, and then is calculated analogously to the 1^{st} solution way, where the solution is found for the *reciprocal* $\frac{1}{x}$:

$$\begin{aligned}
a + \frac{b}{x} + \frac{c}{x^2} &= 0 && \Rightarrow \\
\frac{1}{x} &\;\rightarrow\; y - \frac{b}{2\,c} && \Rightarrow \\
a + b\left(y - \frac{b}{2\,c}\right) + c\left(y^2 - \frac{b}{c}\,y + \frac{b^2}{4\,c^2}\right) &= c\,y^2 - b\,y + b\,y + \frac{b^2}{4\,c} - \frac{b^2}{2\,c} + a = 0 && \Leftrightarrow \\
y^2 &= \frac{b^2}{4\,c^2} - \frac{a}{c} && \Leftrightarrow \\
y &= \pm\sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}\,.
\end{aligned}$$

After resubstituting, the reciprocal is built, where roots in the denominator of a fraction are often transported by corresponding expansion into its numerator:

$$\begin{aligned}
\frac{1}{x} &= -\frac{b}{2\,c} \pm \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}} \qquad \Leftrightarrow \\
x &= \frac{1}{\left(-\frac{b}{2\,c} \pm \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}\right)} \frac{\left(-\frac{b}{2\,c} \mp \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}\right)}{\left(-\frac{b}{2\,c} \mp \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}\right)} = \frac{-\frac{b}{2\,c} \mp \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}}{\frac{b^2}{4\,c^2} - \left(\frac{b^2}{4\,c^2} - \frac{a}{c}\right)} = \\
x &= \frac{c}{a}\left(-\frac{b}{2\,c} \mp \sqrt{\frac{b^2}{4\,c^2} - \frac{a}{c}}\right) = -\frac{b}{2\,a} \mp \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\,. \qquad (17)
\end{aligned}$$

The signs before the *square root* of the solution (17) are swapped in comparison with solution (16). This circumstance emphasizes the difference between the solution ways. Since the numbering of both roots is arbitrary, nevertheless the solutions can be compared to each other.

2.2.4 3rd Solution Way

As 3rd solution way presents itself the *quadratic completion*, which does not need a substitution, but its generalization is difficult only.

$$\begin{aligned}
a\,x^2 + b\,x + c &= 0 \qquad \Leftrightarrow \\
a\left(x^2 + \frac{b}{a}\,x + \frac{b^2}{4\,a^2} - \frac{b^2}{4\,a^2}\right) + c &= 0 \qquad \Leftrightarrow \\
\left(x + \frac{b}{2\,a}\right)^2 &= \frac{b^2}{4\,a^2} - \frac{c}{a} \qquad \Leftrightarrow \\
x + \frac{b}{2\,a} &= \pm\sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}} \qquad \Leftrightarrow \\
x &= -\frac{b}{2\,a} \pm \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\,. \qquad (18)
\end{aligned}$$

2.2.5 Checking Calculation

The 3 solution ways lead to the same solution (16), (17), and (18). The checking calculation in the initial equation (15) can be done for both square roots at once:

$$\begin{aligned}
a\left(-\frac{b}{2\,a} \pm \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\right)^2 + b\left(-\frac{b}{2\,a} \pm \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\right) + c &= 0 \qquad \Leftrightarrow \\
a\left(\frac{b^2}{4\,a^2} \mp \frac{b}{a}\sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}} + \frac{b^2}{4\,a^2} - \frac{c}{a}\right) + b\left(-\frac{b}{2\,a} \pm \sqrt{\frac{b^2}{4\,a^2} - \frac{c}{a}}\right) + c &= 0 \qquad \Leftrightarrow \\
0 &= 0\,. \qquad (19)
\end{aligned}$$

This needed to be shown[22].

[22] Latin version: *quod erat demonstrandum.*

2.3 Calculation of Square Roots

2.3.1 Return to the 4 Basic Arithmetical Operations

Although the *square roots* lead out of the set of the *number fractions*, their numerical calculation is always possible via the 4 basic arithmetical operations. This is very interesting, if only a calculation machine for the 4 basic arithmetic operations[23] is available to calculate the numerical value. The method uses the following connection:

$$z = (10\,a + b)^2 = 100\,a^2 + 20\,a\,b + b^2\,. \tag{20}$$

Here, a is each already known numeral sequence[24] of the square root $\sqrt{z}$, and b is the next following decimal digit. From equation (20) can be seen, that the unknown digit b can be determined the following:

$$\left(z - 100\,a^2\right) : (20\,a) \geq b\,. \tag{21}$$

- In relation (21) is valid $b > 0$, if after subtracting of the already known part a^2 is remaining a rest $z - 100\,a^2 > 0$.
- If a rest $z - 100\,a^2 < 0$ remains, then b is as long to be decreased by unity, until the new rest no longer is negative.
- If the rest is $z - 100\,a^2 = 0$, then the square root is found correctly and can be completed by corresponding, yet missing zeros to the final result.

Now, this method is demonstrated by 3 instructive examples:

2.3.2 Example $\sqrt{729}$

The written calculation of the *square root* causes the following calculation steps:

$$\begin{array}{rcll}
\sqrt{729} & = & 27 & \\
\underline{-4} & = & -a^2 & \Rightarrow a = 2 \\
329 & : & (20\cdot 2) = 8, \ldots & \Rightarrow a = 2\,,\, b = 8 \\
-320 & = & -20\cdot a\cdot b & \\
\underline{-64} & = & -b^2 & \\
\underline{-55} & < & 0 & \Rightarrow a = 2\,,\, b = 7 \\
329 & & \text{repetition} & \\
-280 & = & -20\cdot a\cdot b & \\
\underline{-49} & = & -b^2 & \\
0 & = & 0 & \Rightarrow \text{finish!}
\end{array}$$

The checking calculation yields a confirmation of the result:

$$\begin{array}{l}
\underline{\;27\cdot 27 =\;} \\
\;\;54 \\
\underline{\;\;189\qquad\;} \\
= 729
\end{array}$$

This needed to be shown.

[23] for example a *Chinese abacus*

[24] without decimal point

2.3.3 Example $\sqrt{5}$

Here, after sufficient calculation steps is to be rounded, because $\sqrt{5}$ is no fraction:

$$
\begin{array}{rcll}
\sqrt{5} & = & 2,2360679 & \approx\ 2,236068 \\
\underline{-4} & = & -a^2 & \Rightarrow\ a\ =\ 2 \\
100 & : & (20\cdot 2) = 2,5 & \Rightarrow\ a\ =\ 2\ ,\ b\ =\ 2 \\
-80 & = & -20\cdot a\cdot b & \\
\underline{-4} & = & -b^2 & \\
1600 & : & (20\cdot 22) = 3,\ldots & \Rightarrow\ a\ =\ 22\ ,\ b\ =\ 3 \\
-1320 & = & -20\cdot a\cdot b & \\
\underline{-9} & = & -b^2 & \\
27100 & : & (20\cdot 223) = 6,\ldots & \Rightarrow\ a\ =\ 223\ ,\ b\ =\ 6 \\
-26760 & = & -20\cdot a\cdot b & \\
\underline{-36} & = & -b^2 & \\
\underline{30400} & : & (20\cdot 2236) = 0,\ldots & \Rightarrow\ a\ =\ 2236\ ,\ b\ =\ 0 \\
3040000 & : & (20\cdot 22360) = 6,\ldots & \Rightarrow\ a\ =\ 22360\ ,\ b\ =\ 6 \\
-2683200 & = & -20\cdot a\cdot b & \\
\underline{-36} & = & -b^2 & \\
35676400 & : & (20\cdot 223606) = 7,\ldots & \Rightarrow\ a\ =\ 223606\ ,\ b\ =\ 7 \\
-31304840 & = & -20\cdot a\cdot b & \\
\underline{-49} & = & -b^2 & \\
437151100 & : & (20\cdot 2236067) = 9,\ldots & \Rightarrow\ a\ =\ 2236067\ ,\ b\ =\ 9
\end{array}
$$

2.3.4 Example $\sqrt{2}$

Here, after sufficient calculation steps is to be rounded, because $\sqrt{2}$ is no fraction:

$$
\begin{array}{rcll}
\sqrt{2} & = & 1,414213 & \approx\ 1,41421 \\
\underline{-1} & = & -a^2 & \Rightarrow\ a\ =\ 1 \\
100 & : & (20\cdot 1) = 5 & \Rightarrow\ a\ =\ 1\ ,\ b\ =\ 4 \\
-80 & = & -20\cdot a\cdot b & \\
\underline{-16} & = & -b^2 & \\
400 & : & (20\cdot 14) = 1,\ldots & \Rightarrow\ a\ =\ 14\ ,\ b\ =\ 1 \\
-280 & = & -20\cdot a\cdot b & \\
\underline{-1} & = & -b^2 & \\
11900 & : & (20\cdot 141) = 4,\ldots & \Rightarrow\ a\ =\ 141\ ,\ b\ =\ 4 \\
-11280 & = & -20\cdot a\cdot b & \\
\underline{-16} & = & -b^2 & \\
60400 & : & (20\cdot 1414) = 2,\ldots & \Rightarrow\ a\ =\ 1414\ ,\ b\ =\ 2 \\
-56560 & = & -20\cdot a\cdot b & \\
\underline{-4} & = & -b^2 & \\
383600 & : & (20\cdot 14142) = 1,\ldots & \Rightarrow\ a\ =\ 14142\ ,\ b\ =\ 1 \\
-282840 & = & -20\cdot a\cdot b & \\
\underline{-1} & = & -b^2 & \\
10075900 & : & (20\cdot 141421) = 3,\ldots & \Rightarrow\ a\ =\ 141421\ ,\ b\ =\ 3
\end{array}
$$

The checking calculations with the rounded results confirm quite precisely the calculation method.

2.3.5 Iteration due to Isaac Newton

Isaac Newton found a further calculation way, which even for complex numbers allows to calculate the root quite simply. To understand this solution way, the *differential calculus* is needed. With this, the *derivative* of a function gives the gradient of the same at the considered point x. This gradient is built as *limit*[25] of a *differential quotient*:

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\,, \tag{22}$$

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x}\,, \tag{23}$$

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x-\Delta x)}{2\,\Delta x}\,. \tag{24}$$

If all 3 variations (22), (23), and (24) at the position x are the same, then the function $f(x)$ is *continuous* at this position x, for the other cases alternative calculation ways are to be used to determine the gradient for a certain direction.
The derivative of the square of a function $f(x)$ yields:

$$\begin{aligned}\left(f(x)^2\right)' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)^2 - f(x)^2}{\Delta x} = \\ &= \lim_{\Delta x \to 0} \left(f(x+\Delta x) + f(x)\right) \frac{f(x+\Delta x) - f(x)}{\Delta x} = 2\,f(x)\,f'(x)\,,\end{aligned} \tag{25}$$

$$\begin{aligned}\left(f(x)^2\right)' &= \lim_{\Delta x \to 0} \frac{f(x)^2 - f(x-\Delta x)^2}{\Delta x} = \\ &= \lim_{\Delta x \to 0} \left(f(x) + f(x-\Delta x)\right) \frac{f(x) - f(x-\Delta x)}{\Delta x} = 2\,f(x)\,f'(x)\,,\end{aligned} \tag{26}$$

$$\begin{aligned}\left(f(x)^2\right)' &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)^2 - f(x-\Delta x)^2}{2\,\Delta x} = \\ &= \lim_{\Delta x \to 0} \left(f(x+\Delta x) + f(x-\Delta x)\right) \frac{f(x+\Delta x) - f(x-\Delta x)}{2\,\Delta x} = \\ &= 2\,f(x)\,f'(x)\,.\end{aligned} \tag{27}$$

The derivative of x yields:

$$x' = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1\,, \tag{28}$$

$$x' = \lim_{\Delta x \to 0} \frac{x - (x - \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1\,, \tag{29}$$

$$x' = \lim_{\Delta x \to 0} \frac{(x + \Delta x) - (x - \Delta x)}{2\,\Delta x} = \lim_{\Delta x \to 0} 1 = 1\,. \tag{30}$$

Analogously follows the derivative of a constant $y = \left(\sqrt{y}\right)^2$, being independent of x, where now the *Leibniz notation* is necessary to build the correct limit:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{y - y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y - y}{2\,\Delta x} = 2\sqrt{y} \lim_{\Delta x \to 0} \frac{\sqrt{y} - \sqrt{y}}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0\,. \tag{31}$$

Also this result can be received by three calculation ways, which distinguish from each other.

[25] Latin: *limes*

To get the zero position x_N of the equation $f(x) - y = 0$, *Newton* calculates the following *iteration*[26]:

$$x_{n+1} = x_n - \frac{f(x_n) - y}{f'(x_n)}\,, \tag{32}$$

$$\sqrt{y} = x_N = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n - \frac{x_n^2 - y}{2\,x_n}\right) = \lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{y}{2\,x_n}\right). \tag{33}$$

Therefore here, the calculation method arranges, that both summands are about equal sized to break the iteration yet before $n = \infty$ to get the result in very good approximation.

2.3.6 Example $\sqrt{729}$

For the case $\sqrt{729}$ results with the starting value $x_0 = 1$, what can also be calculated by a pocket calculator for accountancy[27]:

$$\begin{aligned}
x_1 &= \frac{1}{2} + \frac{729}{2} = 365\\
x_2 &= \frac{365}{2} + \frac{729}{730} = 183,49863\\
x_3 &= 93,735706\\
x_4 &= 50,756446\\
x_5 &= 32,559577\\
x_6 &= 27,474651\\
x_7 &= 27,004100\\
x_8 &= 27,000000\\
x_9 &= 27
\end{aligned} \tag{34}$$

Here, the difference of the last iteration steps is even zero, therefore the solution has been found exactly.

As third solution way the root of the *reciprocal* $\frac{1}{y}$ presents itself, this leads with the starting value $x_0 = 1$ to the following *iteration* due to *Newton*:

$$\begin{aligned}
\frac{1}{\sqrt{y}} &= x_N = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n - \frac{x_n^2 - \frac{1}{y}}{2\,x_n}\right) = \lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{1}{2\,x_n\,y}\right).\\
x_1 &= 0,5006859\\
x_2 &= 0,2517128\\
x_3 &= 0,1285812\\
x_4 &= 0,0696248\\
x_5 &= 0,0446633\\
x_6 &= 0,0376881\\
x_7 &= 0,0370427\\
x_8 &= 0,0370370\\
x_9 &= 0,0370370 \approx \frac{1}{27}
\end{aligned} \tag{35}$$

[26]repetition, see [1987BSGZZ], section 7.1.2.3., page 744–745

[27]without square root function, but with memory capacity

2.3.7 Example $\sqrt{5}$

For the case $\sqrt{5}$ results with the starting value $x_0 = 1$, what can also be calculated by a *table calculation program*[28]:

$$\begin{aligned} x_1 &= \frac{1}{2} + \frac{5}{2} = 3 \\ x_2 &= \frac{3}{2} + \frac{5}{6} = 2,3333333 \\ x_3 &= 2,2380952 \\ x_4 &= 2,2360689 \\ x_5 &= 2,2360680 \\ x_6 &= 2,2360680 \approx \sqrt{5} \end{aligned} \tag{36}$$

Here, the difference of the last iteration steps is almost zero, therefore the solution has been found as a good approximation.
As third solution way the root of the *reciprocal* $\frac{1}{y}$ presents itself, this leads with the starting value $x_0 = 1$ to the following *iteration* due to *Newton*:

$$\begin{aligned} \frac{1}{\sqrt{y}} &= x_N = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n - \frac{x_n^2 - \frac{1}{y}}{2\,x_n} \right) = \lim_{n\to\infty} \left(\frac{x_n}{2} + \frac{1}{2\,x_n\,y} \right) . \\ x_1 &= 0,6 \\ x_2 &= 0,4666667 \\ x_3 &= 0,4476190 \\ x_4 &= 0,4472138 \\ x_5 &= 0,4472136 \\ x_6 &= 0,4472136 \approx \frac{1}{\sqrt{5}} \end{aligned} \tag{37}$$

2.3.8 Example $\sqrt{2}$

For the case $\sqrt{2}$ results with the starting value $x_0 = 1$, what can also be calculated by an own calculation program with the wanted accuracy:

$$\begin{aligned} x_1 &= \frac{1}{2} + \frac{2}{2} = 1,5 \\ x_2 &= 1,4166667 \\ x_3 &= 1,4142157 \\ x_4 &= 1,4142136 \\ x_5 &= 1,4142136 \approx \sqrt{2} \end{aligned} \tag{38}$$

Here, the difference of the last iteration steps is almost zero, therefore the solution has been found as a good approximation.

[28] like *Microsoft Excel*

As third solution way the root of the *reciprocal* $\frac{1}{y}$ presents itself, this leads with the starting value $x_0 = 1$ to the following *iteration* due to *Newton*:

$$\begin{aligned}
\frac{1}{\sqrt{y}} &= x_N = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n - \frac{x_n^2 - \frac{1}{y}}{2\,x_n} \right) = \lim_{n\to\infty} \left(\frac{x_n}{2} + \frac{1}{2\,x_n\,y} \right) . \\
x_1 &= 0,75 \\
x_2 &= 0,7083333 \\
x_3 &= 0,7071078 \\
x_4 &= 0,7071068 \\
x_5 &= 0,7071068 \approx \frac{1}{\sqrt{2}}
\end{aligned} \tag{39}$$

2.3.9 What is Going on?

Now, by each 3 calculation ways the linear and the quadratic equation has been solved. For the calculation of a *square root* 3 examples in each 3 divers calculation ways have been presented.

Many centuries can occur in history of mathematics between a solution and its completion by a second and third solution way. The here presented order of the calculation methods is not always strictly historical, but rather didactically optimized, where background knowledge can be very helpful—like for example at an university.

In the following chapter now not the cubic equations are dealt with, but the *arithmetic sequences*, which at the end motivated the *difference quotients*, the limit of which then was lead by *Newton* and *Leibniz* to the derivative. Newton argued for a long time with Leibniz about the question, who of both had founded the differential and integral calculus. The possibility, that both parallely and independent of each other reached the same results and completed them wonderfully by this, was not considered in that time. Research leads to knowledge, this is the meaning of it.

This situation is similar, as if two *first climbers* meet at the summit. In this case, both greet each other even before the walk to the summit and ask the other one about the difficulty of his trip. Then it makes sense, if the one, whose route has been easier, enters the summit first and by this is the *first climber*. After this, both go down the easier trip, by which the other has managed a *first cross*. The probability, that 3 first climbers meet at the same time at a summit, is very low. Also then, a quarrel can be avoided, if the more wise ones do without.

3 Difference Quotient and Arithmetic

3.1 Difference Quotients

3.1.1 Geometric Sequence

Already the *ancient Greeks* knew the *geometric sequence*. It is the following sum:

$$\sum_{\mu=0}^{n} x^{\mu} \quad = \quad 1 + x + x^2 + x^3 + \ldots + x^n \ = \ ? \tag{40}$$

The solution of this task succeeds at the end, where the following proof via a *telescope sum*[29] is very impressive:

$$\begin{aligned} \Big(x^n + \ldots + x^3 + x^2 + x + 1\Big)\ (x-1) \quad &= \quad x^{n+1} + (x^n - x^n) + \ldots + (x - x) - 1\,, \qquad \Leftrightarrow \\ \sum_{\mu=0}^{n} x^{\mu} \quad &= \quad \frac{x^{n+1} - 1}{x - 1}\,. \end{aligned} \tag{41}$$

The result (41) belongs already to the *theorems*[30], which are not at once plausible without the knowledge of a solution way. Well–known is the discussion of the *ancient Greeks*, whether for $n \to \infty$ a *limit* exists, that is less than infinity. For the case no and $x < 1$ the ancient Greeks constructed a task, due to which a rapid runner would not pass a slow turtle, if he would reach the starting point of the turtle later on.

3.1.2 Generalized Geometric Sequence

The result (41) can be generalized, and then it represents the following *telescope sum*:

$$\begin{aligned} a^{n+1} - b^{n+1} \quad &= \quad a \sum_{\mu=0}^{n} a^{\mu}\, b^{n-\mu} \ - \ b \sum_{\mu=0}^{n} a^{\mu}\, b^{n-\mu}\,, \qquad \Leftrightarrow \\ \sum_{\mu=0}^{n} a^{\mu}\, b^{n-\mu} = \sum_{\mu=0}^{n} a^{n-\mu}\, b^{\mu} \quad &= \quad \frac{a^{n+1} - b^{n+1}}{a - b} = \frac{b^{n+1} - a^{n+1}}{b - a}\,. \end{aligned} \tag{42}$$

Therefore, the terms a and b can be swapped in the generalized geometric sequence (42). For $a = b$ results the limit of a *difference quotient*, for example here of the power function a^{n+1} für $b \to a$, which therefore is the first derivative of the power function a^{n+1} to the base a.
The swapping of a and b may be valid to be a second solution way, like at the proof, that $1 + 1 = 2$ is valid, also the summands 1 can be swapped without a change of the result. The *swapability* of the arguments is a special property of sum and product, which can be discovered by the search for further solution ways.

[29] In the midth occur equal pairs with sum zero, thus the sum is pushed together, except for the first and the last term.

[30] mathematical proposition

3.1.3 A 3$^{\text{rd}}$ Solution Way

In theorem (42) occurs an integer number n, thus as third proof presents itself the so–called *complete induction*:

- First a starting value n_1 is to be found. Here presents itself, to search for the integer number $n_0 = (n_1 - 1)$, for which the equation is *not* fulfilled, while it is fulfilled for n_1. This n_1 is the *induction begin*.
- For the conclusion from n to $(n+1)$ is tried to scribe the terms of the equation for $(n+1)$ to terms, which contain one hand side of the equation for n.
- In the so–called *induction step* the term is inserted from the equation to be proven, which stand on the other hand side of the equation.
- In case of success to show, that this equation is fulfilled, the theorem is valid for integer $n \geq n_1$.
- This needed to be shown.

Thus the proof begins by showing, that the *empty sum* for $n_0 = -2$, $a \neq b$, and $a \neq 0 \neq b$ fails, while it fulfills the equation (42) for $n_1 = -1$:

$$\sum_{\mu=0}^{-2} a^{\mu}\, b^{n-\mu} = 0 \neq \frac{a^{-2+1} - b^{-2+1}}{a-b} = \frac{\frac{1}{a} - \frac{1}{b}}{a-b} = \frac{\frac{b-a}{a\,b}}{a-b} = -\frac{1}{a\,b}\,. \tag{43}$$

$$\sum_{\mu=0}^{-1} a^{\mu}\, b^{n-\mu} = 0 = \frac{a^{-1+1} - b^{-1+1}}{a-b} = \frac{1-1}{a-b} = 0\,. \tag{44}$$

Then, the very induction takes place by the *induction step*:

$$\begin{aligned}\sum_{\mu=0}^{n+1} a^{\mu}\, b^{n+1-\mu} &= a^{n+1} + b \sum_{\mu=0}^{n} a^{\mu}\, b^{n-\mu} = a^{n+1} + b\,\frac{a^{n+1} - b^{n+1}}{a-b} = \\ &= \frac{a^{n+1}\,(a-b) + b\,\left(a^{n+1} - b^{n+1}\right)}{a-b} = \frac{a^{n+2} - b^{n+2}}{a-b}\,.\end{aligned} \tag{45}$$

The right hand side of equation (42) is confirmed for $(n+1)$ by the induction (45). Therefore, this equation is valid for all integer $n \geq -1$ und $a \neq b$.
This needed to be shown.

3.1.4 Crossing to the Geometric Sequence

From the generalized geometric sequence (42) follows the geometric sequence (41), not only for $b = 1$, but also via division by a^n, or b^n. If a and b are integers, then a quotient $q = \frac{a}{b} \neq 1$ or $q = \frac{b}{a} \neq 1$ is built, because for $a \neq b$ is valid:

$$\frac{\sum_{\mu=0}^{n} a^{\mu}\, b^{n-\mu}}{b^n} = \sum_{\mu=0}^{n} q^{\mu} = \frac{a\,\left(\frac{a}{b}\right)^n - b}{a-b} = \frac{q^{n+1} - 1}{q-1} = \frac{1 - q^{n+1}}{1-q}\,. \tag{46}$$

The latest *Leonhard Euler* has introduced geometric sequences with the quotient q and has even carried out the *difference quotients* of arbitrary functions analogously. Therefore, this field is known until totay as the so–called *q–analysis*.

3.2 Arithmetic

3.2.1 Power Function

All authors of mathematical textbooks agree, that the oldest and easiest difference quotients are the same of the power function x^n for integer $n \geq 0$ of n factors x. Here results, that for $n = 0$ the difference quotients are always zero, because $x^0 = x^{(1-1)} = \frac{x}{x} = 1$ is valid, to be precise, for all x.
For $n = 1$ follow all real integer numbers as *arithmetical* sequence of 1^{st} degree, where their *difference quotient* is always unity.

3.2.2 Square Numbers

For $n = 1$ occur properties of the square numbers, which are to be considerable:

0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	
2	2	2	2	2	2	2	2	2	2	2	2	2	2		

Here, $\Delta x = 1$ was chosen. However, this is not yet all, what is possible.
So, for $\Delta x = 2$ result the following, arithmetical difference quotients:

0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	
		2	2	2	2	2	2	2	2	2	2	2	2		

For $\Delta x = 3$ follow already known, arithmetical difference quotients:

0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
	3	5	7	9	11	13	15	17	19	21	23	25	27		
		2	2	2	2	2	2	2	2	2	2				

For $\Delta x = 4$ result the following, arithmetical difference quotients:

0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
		4	6	8	10	12	14	16	18	20	22	24	26		
				2	2	2	2	2	2	2	2				

Here, the difference quotiens of 1^{st} degree complete to the set of the integer numbers. With it, the difference quotients of 1^{st} degree lead for odd Δx to the odd numbers, for even Δx to the even numbers. Since in arithmetic only differences and not difference quotients are considered, for the differences with even Δx result gaps.
The difference quotients of the even square numbers lead for integer Δx to all integer numbers. The arithmetical sequences of the integer square numbers lead for integer Δx to gaps, because in this case for $\Delta x = 2$ occur differences only, that can be divided by $2^2 = 4$.
In history of mathematics, it took a long time, until the *arithmetic* was replaced by the *difference quotients*. A reason for this hesitation may be, that since the *Pythagoras' theorem* the search for integer examples could be systematized by *arithmetic*, while the *difference quotients* help rather less for this. So, the following examples are found for the *Pythagoras' theorem*:

$\Delta x = 1$: $5^2 - 4^2 = 3^2$ $13^2 - 12^2 = 5^2$ $25^2 - 24^2 = 7^2$ $41^2 - 40^2 = 9^2$

$\Delta x = 2$: $5^2 - 3^2 = 4^2$ $10^2 - 8^2 = 6^2$ $17^2 - 15^2 = 8^2$ $26^2 - 24^2 = 10^2$

Here, important is just the insight, that $\Delta x = 1$ does *not* yield all integer examples for the *Pythagoras' theorem*. For the construction of a *rectangular triangle*, the edge ratios 3:4:5 are already handed down by the *ancient Egypts*.

3.2.3 Cubic Numbers

For the square numbers has resulted a systematics for the difference quotients:

$$\frac{(x+\Delta x)^2 - x^2}{\Delta x} \;=\; 2\,x + \Delta x\,. \tag{47}$$

This has lead to the result, that for odd Δx the difference quotient of square numbers reaches all odd numbers and for even Δx all even numbers. Nevertheless, here for the differences of two, even square numbers occur also impossibilities, for example to reach the result 2 or 6. For the *cubic numbers* there are already more complicated situations, thus it is much more hard to reach a wanted number:

$$\frac{(x+\Delta x)^3 - x^3}{\Delta x} \;=\; 3\,x^2 + 3\,x\,\Delta x + \Delta x^2\,. \tag{48}$$

For this a systematics is not yet finished, for example to get a positive integer cubic number by the difference of two, positive integer cubic numbers. Time and again, *Pierre de Fermat* claimed to be able to proof this connection for integer $n > 2$, but his historical proof cannot be found anywhere. *Paul Wolfskehl*[31] was anyhow hindered by this problem to commit suicide. As thank for this nature of the task he founded in 1908 a huge amount of money for the same, who would have proven this theorem until 2007. In 1993, *Andrew Wales* brought forth a proof of about 200 pages length, which after at least one correction is regarded to be consistent, and received the prize money.
The difference between *geometric sequence* and *binomial theorem* indeed becomes clear since the *cubic numbers*. It may be, that *Fermat*, who together with *Blaise Pascal* formulated the *binomial theorem* in its final version, aimed to this. Therefore the difference of two cubic numbers due to equation (42) yields:

$$\frac{a^3 - b^3}{a - b} \;=\; a^2 + a\,b + b^2 \;\neq\; a^2 + 2\,a\,b + b^2 \;=\; (a+b)^2\,. \tag{49}$$

The result (49) suggests the supposition, that the difference of two, neighbored, integer cubic numbers could not be an integer square number. However, exactly to this there is at least one example to the contrary[32]:

$$\frac{8^3 - 7^3}{8 - 7} \;=\; 8^2 + 8\cdot 7 + 7^2 \;=\; 169 \;=\; 13^2 \;=\; (7+6)^2 \;=\; 7^2 \;+\; 2\cdot 7\cdot 6 + 6^2\,. \tag{50}$$

Therefore, the failure of a wanted calculation way does not proof at all the general non–solvability, although the correctness of the inequality (49) is valid for $a \neq 0$ and $b \neq 0$. If such a square number would be also a cubic number, what is the case for all 6^{th} powers of a number, then the claim of Fermat and the proof of Andrew Wiles would be shaken by this. Kurt Gödel categorized Fermat's problem in 1931 to be *undecidable*[33], he leaved it open, who was right. Unfortunately, he and others thought, he had proven the undecidability, thus just a new concept for proven unsolvability of a problem was introduced. However, undecidability can never be proven, but expresses, that currently no solution is available.

[31] [1953VEB], entry *Fermat*, page 293
[32] [1992PRS], page 21
[33] [1931Göd]

3.2.4 Undecidability

The big problems of life are all not solved during a single day. Among them are principle questions of the following kind:

- Is there a God?
- How old is the earth?
- Has someone fallen in love to me?
- Is the food, that I eat, with poison?

Usually, these questions are considered to be *currently undecidable* and sometimes left for later researchers, who will eventually yield the goal:

- The last mentioned question was the daily problem of Kurt Gödel, who by this found each day an undecidability. As long as his dear wife was living, she always knew an argumentation, which moved him to eat the meal, that she had cooked for him. Then, when she died, Kurt Gödel died of starvation because of the *undecidability*, being unsolvable to him. This shows, that for him his research results were real. Therefore, medical doctors time and again call mathematicians to be ill with *obsessive–compulsive disorder* (OCD).
- In connection to falling in love often enough the reason is missing. Therefore, at least in Germany there is an old tradition to find out the very state of the things by pulling a flower to pieces. With this is alternately said at each petal:
  - *"You love me."*
  - *"You do not love me."*

  By this at the end there will indeed be a result, but whether it describes a reality, remains open. Three–leaved clover or four–leaved cruciferous plants are usually excluded from this questioning. Patience and prudence help much better in such questions. Also the advice of parents and friends will protect here from bad luck.
- The question, how old the earch is, has already occupied many humans, which all are younger than the same. Merely in the Bible of the people Israel there are at least 3 variants of written records for Genesis 5, from which only the text version of the *Samaritans*, being yet despised by the Jews, confirms the number value of the *Israelian calendar*, which is used until today. The theologian *Adolf Schlatter* in Tübingen (Germany) considered this question from his sight to be unsolvable, but he permitted, that later on someone else would solve this problem, for example a mathematician. Therefore, he let the 3 divers number columns in German language to be printed in the Calw's Bible Encyclopaedia[34] to bring to an end the hurdle of a Hebrew and Greek study for solving the problem. This was his contribution to the solution of this task. The author needed in spite of this help in total 40 years to clarify the concerning problem finally and offers the result to all interested ones[35]. In each *science*, correct results are offered only, and they are not forced upon anyone.
- The question, whether there is a God, has already been asked so much, that in Germany the legislator meanwhile tends towards ideological and religious *tolerance*: Anyone is

[34] [1924ZH], entry *Seth*, page 699
[35] for example: [2018SW]

allowed to find his own answer to this question. In Germany, the trial to force others to join, converse to, or leave a denomination, is considered to be a violation against the *religious freedom*[36], which is established as an immediately valid, fundamental right. Only for children, the *legal guardians* are permitted to prescribe the belonging to a denomination or to participate in religious instruction. Due to article 136 of the Weimar's constitution[37], no civil rights or duties result automatically by *religious freedom*. Thus, due to the word of *Frederick the Great*, in Germany "anyone shall become blessed according to his version."

Concerning the problem, whether there is a God, these legal frame conditions contribute so much only, that about this *no quarrel* is allowed. Contentionally, this question can be compared with the question about the existence of the electrical current: It is existent also in case of no counting on it.

Whosoever wants to meet the living God, the same should fit in with his frame conditions. Already in the Holy Bible of the people Israel, which reports the clearest on such meetings, the following statements[38] are found:

– *But without faith it is impossible to please him: for he that comes to God must believe that he is, and that he is a rewarder of them that diligently seek him.* (Hebrews 11:6)
– *Because that which may be known of God is manifest to them; for God hath shown it unto them.*
  For the invisible things of him from the creation of the world are clearly seen, being understood by the things that are made, even his eternal power and Godhead; so that they are without excuse. (Romans 1:19–20)
– *And that we may be delivered from unreasonable and wicked men: for all men have not faith.* (2nd Thessalonians 3:2)

Therefore it is beneath the dignity of Israel's God, to discuss with humans about his existence. In the tradition of the Roman Catholic church, the cited text from the Roman's epistle was often quoted, but explained seldom with the help of examples, thus not yet nearly anyboby will find by the creation to the creator, too. Also in the Swabian pietism of the evangelical tradition, there is often use to get out of the way of these questions by pious excuse, instead of consenting to take part in the problem, and to answer at least partially.

On the other hand side, there are also deceitful mockers, which even in denominations speak only to cause confusion. So, the French mathematician and philosopher *Pierre Simon de Laplace* became well known for the following answer to Napoleon: *"Majesty, the hypothsis 'God' I do not need."* This comment shows, that he indeed had read many philosophers, but instead of a present *undecidability* he tended to mocking. To the wisdom of Israel's God belongs, that also his existence he does not force to any human. Therefore, the Holy Bible totally refrains philosophical proofs of the existence of God.

[36] [2001GG], article 4, page 14–15
[37] [2001GG], article 140, page 85 and 89
[38] each cited from [1994AV] with adapted spelling

3.2.5 The End of the Fundamental Crisis

Albert Einstein and *Kurt Gödel* became friends, after his *undecidability theorem* had become public knowledge. Both were searching for ways out of the scholastic dogmatizing of *Aristoteles*, which is tradition since the Middle Ages. Due to Aristoteles, to a question there would only be the answers[39] *"right"* or *"wrong"*. But, that even the important problems of life lead first into an undecidability, was insensitively ignored by the cite: *"tertium non datur."*[40] *Gödel* however found examples for *undecidability*, among which he restricts himself in his elaboration[41] to mathematical problems:

- The supposition of *Fermat*.
- The set theory, which also can yield partial congruence, where the opposite of which is another partial congruence.
- The analytical solvability of the algebraic equations of 5^{th} and higher degree.

That privately he was occupied each day by the *undecidability*, whether his meal would be poisoned, may have been a later occuring problem. Indeed, Gödel was very logical and powerless in view of undecidability.
The scholastics also yet today ignore the solution suggestions by Gödel and others and claim, that by his work a fundamental crisis has befallen mathematics. This crisis exists only for philosophers, which want to proof imperatively, instead of searching for *coincidence* of several solution ways. Whosoever wants to replace the word *"partial congruence"* by a good, English word, the same shall use the word *"possible"*. The opposite of this *"possible"* now is another *"possible"* and by no means *"impossible"*. This means, that by this expansion of the *Boolean algebra* an alternative thinking can begin in mathematics, which in the long term will save from *obsessive–compulsive disorder*. Since *Aristoteles* was a human, also he is allowed to once have been mistaken: *tertium datur*[42].
By Gödel's undecidability the notion unsolvability is banished from mathematics: On unsolvability can decide only he, who knows and has tried all available solution possibilities. Even his failing does not proof, that other researchers will also fail with this problem. This is like a *first climb* of a summit: As long as nobody was upon, there are indeed research approaches, but not yet successful ones. The existence of the summit is not shaken by this, even if it just stands in the clouds.
Therefore, whosoever wants to be saved from the scholastic dogmatizing of *Aristoteles* since the Middle Ages, the same shall search further alternatives for each solution way. This possibility of self–check saves each researcher very effectively from all kinds of fallacies, but it is significantly more laborious, than the repeated parrot–fashion of theorems, being learned by heart, plus their supposedly only possible derivation.
Concerning these thoughts, now the author does not present each 3 derivations, but rather wants to prompt to independent thinking, searching and working:

> *No mountain guide carries his guests onto the summit,*
> *but everybody is allowed to climb and clamber one'self.*

[39] also known as *Boolean algebra*
[40] Latin for: *"A third one has not been given."*
[41] [1931Göd]
[42] Latin for: *"A third one has been given."*

3.2.6 Chinese Arithmetic

The *binomial theorem* detects the coefficients, which result by expanding multiplication of the $n^{\rm th}$ power of a sum $(a+b)$:

$$(a+b)^n \quad = \quad \sum_{\mu=0}^{n} \binom{n}{\mu} a^\mu \, b^{n-\mu} \,. \tag{51}$$

The *binomial coefficients* $\binom{n}{\mu}$, occuring there, are already found on old, Chinese wood-cuttings[43]. Their number values result from a task requiring great diligence by repeated, expanding multiplication, where also their formation law

$$\binom{n+1}{k+1} \quad = \quad \binom{n}{k} + \binom{n}{k+1} \tag{52}$$

can be found[44]:

n	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0
3	1	3	3	1	0	0	0	0
4	1	4	6	4	1	0	0	0
5	1	5	10	10	5	1	0	0
6	1	6	15	20	15	6	1	0
7	1	7	21	35	35	21	7	1

The puzzling by *Blaise Pascal* at the end has lead to the following formula to calculate the *binomial coefficients* directly[45]:

$$\binom{n}{k} \quad = \quad \frac{n!}{k!\,(n-k)!}\,, \qquad n! \,=\, \prod_{\mu=1}^{n} \mu \,=\, 1\cdot 2\cdot 3\cdots n\,, \qquad 0! \,=\, 1\,. \tag{53}$$

There are again several calculation ways to find such a solution. The most known is done by introduction of a function *factorial* $n!$, which fulfills the following *difference equation* and reduces the difference equation (52) to the following problem:

$$(n+1)! \quad = \quad (n+1)\,n!\,, \qquad 1! := 1 \tag{54}$$

By help of this difference equation (54), the value for 0! can be set at once. Meanwhile the factorial function is tought in the middle level of grammar schools and nevertheless is already the entrance to higher mathematics.
Now, if the Chinese or also Pascal's triangle is layed to the left hand side onto all unities, then results the insight, that especially simple, *arithmetical sequences* lay one above the other, thus *canonical polynomials* of $k^{\rm th}$ degree have been found, for example:

$$\binom{x}{0} \quad = \quad 1\,, \qquad \binom{x}{1} = x\,, \qquad \binom{x}{2} = \frac{(x-1)\,x}{2}\,, \qquad \binom{x}{3} = \frac{(x-2)\,(x-1)\,x}{6}\,. \tag{55}$$

[43] [1995Oli], figure 42, page 103
[44] [1987BSGZZ], section 2.2.1.2., page 104
[45] [1987BSGZZ], section 2.2.1.2., equation (2.1), page 104

3.2.7 Newton's Arithmetic

The lastest *Isaac Newton* changed the *difference sequences of arithmetic* by *arithmetical difference quotients*. The reason for this comes out of physics:

- For mesured data sequences, the measuring interval plays an important role, which not at all is unity only.
- Material properties of steels are often reported in a distance of 100 K.
- The mathematical interpolation of a measured sequence must be independent of the used measuring scale.
- Tidily recorded data sequences have got a throughout constant measuring interval.

So, Isaac Newton did not philosophize the whole life on square and cubic numbers, but generated mathematical tools to cope with the usual research day. His *interpolation formula*[46] is valid only for $(n+1)$ equidistant data steps and reproduces each even great measuring sequence into a *polynomial*, which can be calculated very quickly:

$$f(x) = \sum_{\mu=0}^{n} \binom{\frac{x}{\Delta x}}{\mu} \Delta^\mu f(x)\big|_{x\to x_0} = \sum_{\mu=0}^{n} \binom{\frac{x}{\Delta x}}{\mu} \Delta x^\mu \left(\frac{\Delta^\mu f(x)}{\Delta x^\mu}\bigg|_{x\to x_0} \right), \tag{56}$$

$$f(x) = \sum_{\mu=0}^{n} \binom{\frac{x}{\nabla x}}{\mu} (-1)^\mu\ \nabla^\mu f(x)\big|_{x\to x_0} = \sum_{\mu=0}^{n} \binom{\frac{x}{\nabla x}}{\mu} (-\nabla)^\mu \left(\frac{\nabla^\mu f(x)}{\nabla x^\mu}\bigg|_{x\to x_0} \right). \tag{57}$$

Here, Δ means a difference like in the numerator of the difference quotient (22), and ∇ a difference like in the numerator of the difference quotient (23). From both Newton's formulae (56) and (57) also a third calculation way analogously to (24) can be constructed, yet. In connection to this interpolation, time and again a *rest term* is discussed, because not always was understood, that this interpolation deals with finitely sized, measured data. Of course, by this can also be found very useful things concerning *polynomial differences* or *polynomial sums*:

- The numerical derivative by the derivative of Newton's interpolation has got very lower sized noise, than by other algorithms.
- The numerical integration of measured data can be done directly by use of Newton's interpolation—also for discrete sums.
- The coefficients of the difference sums Δ^μ or ∇^μ are always characteristic of power terms and therefore also of polynomials.

For the difference sums Δ^μ of x^2 result only 2 coefficients and by this all sums[47] and differences of x^2:

$$x^2 = 0\binom{x}{0} + 1\binom{x}{1} + 2\binom{x}{2} = x + 2\,\frac{x\,(x-1)}{2} = x^2\,, \tag{58}$$

$$\sum_{\nu=0}^{x} \nu^2 = \binom{x+1}{2} + 2\binom{x+1}{3} = \frac{x^2+x}{2} + \frac{x^3-x}{3} = \frac{x\,(x+1)\,(2\,x+1)}{6}. \tag{59}$$

The number of coefficients or difference sums Δ^μ in equation (56) is reduced to a minimum, if the same are taken at the beginning by zero.

[46] [1987BSGZZ], section 7.1.2.6.2., table 7.9, page 758

[47] see [1987BSGZZ], section 2.3.3., equation (5), page 114

3.2.8 Heuristics of Variants

When listing and systematizing the coefficients or *difference sums* $\Delta^k x^p$ result several possibilities, which are worth to be mentioned particularly:

Exponent	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
p = 0	1	0	0	0	0	0	0	0
p = 1	0	1	0	0	0	0	0	0
p = 2	0	1	2	0	0	0	0	0
p = 3	0	1	6	6	0	0	0	0
p = 4	0	1	14	36	24	0	0	0
p = 5	0	1	30	150	240	120	0	0
p = 6	0	1	62	540	1560	1800	720	0
p = 7	0	1	126	1806	8400	16800	15120	5040

As building law for the coefficients $K_1(p,k)$ results with the exponent p:

$$K_1(p+1,k+1) \quad = \quad (k+1)\ (K_1(p,k) + K_1(p,k+1))\ . \tag{60}$$

Now, the difference sums can also be listed, beginning at unity[48] and lead by this to a shift of the index μ in equation (56), to yield again by use of Newton's interpolation correct results:

Exponent	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7
p = 0	1	0	0	0	0	0	0	0
p = 1	1	1	0	0	0	0	0	0
p = 2	1	3	2	0	0	0	0	0
p = 3	1	7	12	6	0	0	0	0
p = 4	1	15	50	60	24	0	0	0
p = 5	1	31	180	390	360	120	0	0
p = 6	1	63	602	2100	3360	2520	720	0
p = 7	1	127	1932	10206	25200	31920	20160	5040

As building law for the coefficients $K_2(p,k)$ results[49] with the exponent p:

$$K_2(p+1,k+1) \quad = \quad (k+1)\, K_2(p,k) + (k+2)\, K_2(p,k+1)\,. \tag{61}$$

A further variant is found in literature by the entry *Stirling's numbers of* 2^{nd} *kind*[50], which cannot be presented here because of publisher's rights, and also concerning content causes more confusion than use, because it enables hardly no simple heuristics for arbitrary sums and differences, like at *Newton*. As building law for the coefficients $K_{S,2}(p,k)$ results[51], if again now is chosen $k \geq 0$, instead of else usually $k \geq 1$:

$$K_{S,2}(p+1,k+1) \quad = \quad K_{S,2}(p,k) + (k+2)\, K_{S,2}(p,k+1)\,. \tag{62}$$

Therefore, this difference equation results, if the *binomial coefficients* (55) are replaced by so–called *factorial polynomials*, which miss the division by the belonging factorial $k!$. In return for this, the difference sums $K_2(p,k) \;=\; k!\, K_{S,2}(p,k)$ are divided in the here presented scaling of k by $k!$. Of course, polynomials can be scaled and represented arbitrarily. The mentioned diversity gives an orientation, what is to be taken into account for own calculation programs: There are *several* calculation ways to the goal.

48 [1992PRS], page 12

49 [1992PRS], equation (3), page 9

50 [1982ST], appendix B, page 233

51 [1982ST], equations (22), (27), (31) and (32), page 6–7

4 Unity Roots and Complex Numbers

4.1 Unity Roots due to Gauß

In connection to his doctoral thesis, Carl Friedrich Gauß solved among others the problem, which roots q has got the polynomial $q^n - 1$. For this, he knew because of the *geometric sequence* (41), that a *polynomial division* by $(q-1)$ works always out, as long as n is an integer number. Therefore, in this manner the equation $q^3 - 1 = 0$ can be yet solved by the already presented methods, because the quadratic equation has already been solved in general:

$$\begin{aligned} q^3 - 1 &= (q-1)\left(q^2+q+1\right) = (q-1)\left(q+\frac{1}{2}+\frac{\sqrt{-3}}{2}\right)\left(q+\frac{1}{2}-\frac{\sqrt{-3}}{2}\right) \\ q_1 &= 1\,, \qquad q_2 = -\frac{1}{2}+\frac{\sqrt{-3}}{2}\,, \qquad q_3 = -\frac{1}{2}-\frac{\sqrt{-3}}{2}\,. \end{aligned} \tag{63}$$

All three solutions (63) fulfill the checking calculation in the equation $q^3 = 1$, what results by inserting and expanding multiplication.

4.2 Gaussian Number Plane

If $\sqrt{-1}$ represents an own number dimension, then the 3 solutions (63) build an *equilateral triangle*, one corner of which is placed at the coordinate $\{1;0\}$, and its other corners at the coordinates $\{-\frac{1}{2};\pm\frac{\sqrt{3}}{2}\}$. Thus, the *complex numbers* are interpreted geometric and stretch out the *Gaussian number plane*, while all *real numbers* are hold by a one–dimensional *number beam*. Now, the *absolute value* of the solutions (63) results due to *Pythagoras* as *line* between the discussed point and the coordinate origin:

$$\begin{aligned} \sqrt{1^2+0^2} &= 1 \\ \sqrt{\left(-\frac{1}{2}\right)^2+\left(\pm\frac{\sqrt{3}}{2}\right)^2} &= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1\,. \end{aligned}$$

As a consequence, all roots of unity are on the *unity circle* in the complex number plane. Their absolute value is each unity, they distinguish only by their *phase angle*, which is measured counter–clockwisely from the positive real axis, and by this shall own the rotation orientation *mathematical positive*. By Gauß, only phase angles as *radial arc* are allowed in the range $[0;2\pi)$, by which also the *polar coordinates absolute value* and *phase angle* are always unambiguous. This setting seems to be arbitrary, but it has been calibrated sensibly by Gauß, so that *algebra* can be dealed with clearly and in general best possibly.
The physicist and mathematician *Stephen Wolfram* deviates at his mathematics platform *Mathematica* from the settings due to Gauß, by calculating the phase angle of a complex number also indeed as *radial arc*, however now in the interval range $(-\pi;\pi]$. As a consequence, the formulae for complex numbers can turn out to be hardly understood at *Mathematica*, furthermore the applicants must think and calculate in diverse variants, by which creep in many mistakes, if not the principle of *three calculation ways* is applied. This means:

Mathematica does not calculate wrong in general,
but time and again *differently* than in literature.

However, *Mathematica* allows the own easy programming of traditional complex numbers. This is the real strength of this mathematics platform.

4.3 The Root Theorem

4.3.1 The Formula

The following theorem is fundamental to be able to calculate with complex numbers:

$$\sqrt[n]{a\,b} \;=\; \sqrt[n]{a}\,\sqrt[n]{b}\,. \tag{64}$$

4.3.2 1st Proof

As proof, for integer $n > 0$ the n^{th} power is built as *inverse function* of the n^{th} root with $(\sqrt[n]{c})^n = c$:

$$\begin{aligned}\left(\sqrt[n]{a\,b}\right)^n &= \left(\sqrt[n]{a}\,\sqrt[n]{b}\right)^n = \prod_{\mu=1}^{n}\left(\sqrt[n]{a}\,\sqrt[n]{b}\right) = \left(\prod_{\mu=1}^{n}\sqrt[n]{a}\right)\left(\prod_{\mu=1}^{n}\sqrt[n]{b}\right), \qquad \Leftrightarrow \\ a\,b &= (\sqrt[n]{a})^n\left(\sqrt[n]{b}\right)^n = a\,b\,.\end{aligned}$$

By this, the n^{th} power is proven as *inverse function* for both sides of equation (64).

4.3.3 2nd Proof

With $a = c^n$ and $b = d^n$ follows for integer $n > 0$:

$$\sqrt[n]{a\,b} \;=\; \sqrt[n]{c^n\,d^n} = \sqrt[n]{(c\,d)^n} = c\,d = \sqrt[n]{a}\,\sqrt[n]{b}\,.$$

By this, the root theorem (64) is proven by suitable substitution[52].

4.3.4 3rd Proof

With $a^{-n} = \frac{1}{a^n}$ follows for the reciprocal of equation (64) and integer $n > 0$:

$$\begin{aligned}\left(\frac{1}{\sqrt[n]{a\,b}}\right)^n &= \frac{1^n}{\left(\sqrt[n]{a\,b}\right)^n} = \frac{1}{a\,b} = \frac{1}{(\sqrt[n]{a})^n\left(\sqrt[n]{b}\right)^n} = \left(\frac{1}{\sqrt[n]{a}\,\sqrt[n]{b}}\right)^n \qquad \Leftrightarrow \\ \frac{1}{\sqrt[n]{a\,b}} &= \frac{1}{\sqrt[n]{a}\,\sqrt[n]{b}} \qquad \Leftrightarrow \\ \sqrt[n]{a\,b} &= \sqrt[n]{a}\,\sqrt[n]{b}\,.\end{aligned}$$

This needed to be shown.

4.3.5 Outlook

In the frame of further proofs will follow later on the generalization of the root theorem (64) to all complex numbered n. This generalization is the easiest in the phase notation due to Gauß and especially with *Mathematica* needs finishing off. Each interested one may consider one'self, which programming platform would seem to be best suitable for him.

[52] replacement

4.4 Exponential Function

The following limit leads by the substitution $n = m\,x$ to the *exponential function*:

$$\begin{aligned}
\mathrm{e}^x &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m\,x} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \\
&= \lim_{n\to\infty}\sum_{\mu=0}^{n}\binom{n}{\mu}\left(\frac{x}{n}\right)^\mu = \lim_{n\to\infty}\sum_{\mu=0}^{n}\frac{x^\mu}{\mu!}\,\frac{n!}{(n-\mu)!\,n^\mu} = \\
&= \lim_{n\to\infty}\sum_{\mu=0}^{n}\frac{x^\mu}{\mu!}\prod_{k=1}^{\mu}\left(\frac{n-\mu+k}{n}\right) = \sum_{\mu=0}^{\infty}\frac{x^\mu}{\mu!}\prod_{k=1}^{\mu}1 = \sum_{\mu=0}^{\infty}\frac{x^\mu}{\mu!}\,. \qquad (65)
\end{aligned}$$

The found series (65) determines Euler's number e for $x = 1$. An especially fast programming results for $x \geq 0$ by placing the same terms outside the brackets:

$$\mathrm{e}^x = 1+\frac{x}{1}\left(1+\frac{x}{2}\left(1+\frac{x}{3}\left(1+\frac{x}{4}\,(\ldots)\right)\right)\right)\,.$$

Mainly this means, that the sum term of the series is permanently changed by a product, by what a uniform calculation time is to be expected per loop. The derivative yields:

$$\begin{aligned}
\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} &= \lim_{\Delta x\to 0}\frac{\mathrm{e}^{x+\Delta x}-\mathrm{e}^x}{\Delta x} = \mathrm{e}^x\lim_{\Delta x\to 0}\frac{\mathrm{e}^{\Delta x}-1}{\Delta x} = \mathrm{e}^x\lim_{\Delta x\to 0}\lim_{n\to\infty}\frac{\left(1+\frac{\Delta x}{n}\right)^n-1}{\Delta x} = \\
&= \mathrm{e}^x\lim_{\Delta x\to 0}\lim_{n\to\infty}\sum_{\mu=1}^{n}\binom{n}{\mu}\frac{\Delta x^{\mu-1}}{n^\mu} = \mathrm{e}^x\left(1+\lim_{n\to\infty}\sum_{\mu=2}^{n}\binom{n}{\mu}\frac{0}{n^\mu}\right) = \mathrm{e}^x\,. \qquad (66)
\end{aligned}$$

An alternative calculation way is the derivative of the exponential series, for what the derivative of an integer power x^n is needed with $\mu > 0$:

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x} &= \lim_{\Delta x\to 0}\sum_{\mu=1}^{n}\binom{n}{\mu}\Delta x^{\mu-1}\,x^{n-\mu} = \\
&= n\,x^{n-1}+\sum_{\mu=2}^{n}\binom{n}{\mu}0\,x^{n-\mu} = n\,x^{n-1}\,. \qquad (67)
\end{aligned}$$

Also for this result (67), by (23) or (24) there are alternative calculation ways to (22). A further variant results via the generalized geometric sequence (42):

$$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x} = \lim_{\Delta x\to 0}\sum_{\mu=0}^{n-1}(x+\Delta x)^\mu\,x^{n-1-\mu} = \sum_{\mu=0}^{n-1}x^{n-1} = n\,x^{n-1}\,. \qquad (68)$$

Now, by this follows the derivative of the exponential series:

$$\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = \sum_{\mu=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^\mu}{\mu!}\right) = \sum_{\mu=0}^{\infty}\frac{\mu\,x^{\mu-1}}{\mu!} = \sum_{\mu=1}^{\infty}\frac{x^{\mu-1}}{(\mu-1)!} = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!} = \mathrm{e}^x\,. \qquad (69)$$

A third solution way for the derivative of the exponential function results via the derivative of the *inverse function*, namely the natural logarithm $\ln(x)$.

4.5 Natural Logarithm

The *natural logarithm* is the *inverse function* of the *exponential function* (65):

$$\ln\left(e^x\right) \;:=\; x\,, \qquad e^{\ln(x)} := x\,. \tag{70}$$

There are again several possibilities to calculate, of which *Newton's iteration* (32) is very quickly:

$$\begin{aligned} x &= \ln(y)\,, \qquad y = e^x\,, \\ x &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n - \frac{e^{x_n} - y}{e^{x_n}} = \lim_{n\to\infty} x_n - 1 + \frac{y}{e^{x_n}}\,. \end{aligned} \tag{71}$$

During the search for alternative calculation ways, also here the *reciprocal* helps to go on:

$$\begin{aligned} x &= \ln(y)\,, \qquad \frac{1}{y} = e^{-x}\,, \\ x &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n + \frac{e^{-x} - \frac{1}{y}}{e^{-x}} = \lim_{n\to\infty} x_n + 1 - \frac{e^x}{y}\,. \end{aligned} \tag{72}$$

As example presents itself the *natural logarithm* of unity, beginning at $x_0 = 1$:

x_1	$=$	$1 - 1 + \frac{1}{e^1} = 0,3678794$	x_1	$=$	$1 + 1 - \frac{e^1}{1} = -0,7182818$
x_2	$=$	$0,0600801$	x_2	$=$	$-0,2058711$
x_3	$=$	$0,0017692$	x_3	$=$	$-0,0198091$
x_4	$=$	$0,0000016$	x_4	$=$	$-0,0001949$
x_5	$=$	$0,0000000$	x_5	$=$	$-0,0000000$
x	$=$	$0 - 1 + \frac{1}{e^0} = 0\,,$	x	$=$	$0 + 1 - \frac{e^0}{1} = 0\,.$

The following example finds the natural logarithm of 2, each beginning at $x_0 = 0$:

x_1	$=$	$0 - 1 + \frac{2}{e^0} = 1,0000000$	x_1	$=$	$0 + 1 - \frac{e^0}{2} = 0,5000000$
x_2	$=$	$0,7357589$	x_2	$=$	$0,6756394$
x_3	$=$	$0,6940423$	x_3	$=$	$0,6929948$
x_4	$=$	$0,6931476$	x_4	$=$	$0,6931472$
x_5	$=$	$0,6931472$	x_5	$=$	$0,6931472$
x_6	$=$	$0,6931472$			

Because of (70), for the logarithm $\log_b(y)$ to base b is valid:

$$b^x \;=\; e^{x\,\ln(b)} \qquad \log_b\left(b^x\right) := x \qquad \log_b(y) \;=\; \frac{\ln(y)}{\ln(b)}\,. \tag{73}$$

Because of (73), all logarithms are *proportional* to each other. The derivative of the logarithm yields with $\ln(a) + \ln(b) = \ln(ab)$:

$$\begin{aligned} \frac{d\ln(x)}{dx} &= \lim_{\Delta x\to 0} \frac{\ln(x+\Delta x) - \ln(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\ln\left(\frac{x+\Delta x}{x}\right)}{\Delta x} = \\ &= \lim_{\Delta x\to 0} \ln\left(\left(1 + \frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}\right) = \lim_{n\to\infty} \ln\left(\left(1 + \frac{1}{n}\right)^{\frac{n}{x}}\right) \ln\left(e^{\frac{1}{x}}\right) = \frac{1}{x}\,. \end{aligned} \tag{74}$$

If $x = e^y$ is set, then the result (74) follows also by the *derivative of the inverse function*—a connection, which is only valid for the first derivatives and has lead to the *notation due to Leibniz* with *differential fractions*:

$$\frac{d\ln(x)}{dx} \;=\; \frac{dy}{de^y} = \frac{1}{\frac{de^y}{dy}} = \frac{1}{e^y} = \frac{1}{x}\,. \tag{75}$$

4.6 Complex Numbers

Complex numbers have got two real numbers, which can be drawn in two dimensions. As coordinate system present themselves the Cartesian and the polar coordinates. They are *not* completely equivalent, because with distance or absolute value zero an arbitrary phase angle is yet possible, which however cannot be set in Cartesian coordinates for the absolute value zero.
Therefore, for computer algebra mainly are suitable complex numbers in *polar coordinates*, thus the *absolute value* $|z|$ and *phase* $\arg(z)$ as angle with *radial arc* are set, for which the *trigonometric functions* from geometry play a role:

$$\begin{aligned}
z &= \Re(z) + \mathrm{i}\,\Im(z)\,, && (76)\\
\overline{z} &= \Re(z) - \mathrm{i}\,\Im(z)\,, && (77)\\
\Re(z) &= \frac{z+\overline{z}}{2} = |z|\,\cos(\arg(z)) = |\overline{z}|\,\cos(\arg(\overline{z}))\,, && (78)\\
\Im(z) &= \frac{z-\overline{z}}{2\,\mathrm{i}} = |z|\,\sin(\arg(z)) = -|\overline{z}|\,\sin(\arg(\overline{z}))\,, && (79)\\
|z| &= \sqrt{z\,\overline{z}} = \sqrt{\Re(z)^2+\Im(z)^2} = \sqrt{|z|^2\,(\cos(\arg(z))^2+\sin(\arg(z))^2)} = |\overline{z}|\,, && (80)\\
\arg(z) &= \arctan\left(\frac{\Im(z)}{\Re(z)}\right) = \arctan\left(\frac{z-\overline{z}}{z+\overline{z}}\right) = \arctan\left(\frac{|z|\,\sin(\arg(z))}{|z|\,\cos(\arg(z))}\right) = \arg(z)\,, && (81)\\
\arg(\overline{z}) &= \arctan\left(\frac{\Im(\overline{z})}{\Re(\overline{z})}\right) = \arctan\left(\frac{\overline{z}-z}{\overline{z}+z}\right) = \arctan\left(\frac{\sin(\arg(\overline{z}))}{\cos(\arg(\overline{z}))}\right)) = -\arg(z)\,, && (82)\\
z &= |z|\,(\cos(\arg(z)) + \mathrm{i}\,\sin(\arg(z))) = |\overline{z}|\,(\cos(\arg(\overline{z})) - \mathrm{i}\,\sin(\arg(\overline{z})))\,. && (83)
\end{aligned}$$

In the definitions (78) until (81), the number pairs $\Re(z)$ and $\Im(z)$ of the Cartesian coordinates are calculated into the number pairs $|z|$ and $\arg(z)$ of the polar coordinates, if the trigonometric functions are known from geometry.
Now, since *Pythagoras* is known, that in the *unity circle* is valid for the *rectangular triangle*:

$$\sin(x)^2+\cos(x)^2 = 1\,. \tag{84}$$

This insight (84) in equation (83) leads to $\cos(\arg(z))$:

$$\begin{aligned}
\frac{z}{|z|} = \frac{\Re(z)+\mathrm{i}\,\Im(z)}{\sqrt{\Re(z)^2+\Im(z)^2}} &= \cos(\arg(z)) + \mathrm{i}\sqrt{1-\cos(\arg(z))^2} && \Leftrightarrow\\
\frac{z}{|z|} - \cos(\arg(z)) &= \sqrt{\cos(\arg(z))^2-1} && \Leftrightarrow\\
\left(\frac{z}{|z|}\right)^2 - 2\,\frac{z}{|z|}\,\cos(\arg(z)) + \cos(\arg(z))^2 &= \cos(\arg(z))^2-1 && \Leftrightarrow\\
\cos(\arg(z)) &= \frac{\frac{z}{|z|}+\frac{|z|}{z}}{2}\,. && (85)
\end{aligned}$$

Therefore, $\cos(\arg(z))$ is the arithmetic mean of a number $\frac{z}{|z|}$ and its reciprocal. By use of the exponential function (65), this can be written the following:

$$\cos(\arg(z)) = \frac{\mathrm{e}^{\frac{\mathrm{i}\,\ln\left(\frac{z}{|z|}\right)}{\mathrm{i}}} + \mathrm{e}^{-\frac{\mathrm{i}\,\ln\left(\frac{z}{|z|}\right)}{\mathrm{i}}}}{2} = \frac{\mathrm{e}^{\mathrm{i}\,\arg(z)}+\mathrm{e}^{-\mathrm{i}\,\arg(z)}}{2}\,. \tag{86}$$

Analogously follows sin(arg(z)) with $\mathrm{i} = \sqrt{-1}$:

$$\begin{aligned}
\frac{z}{|z|} &= \sqrt{1-\sin(\arg(z))^2} + \mathrm{i}\,\sin(\arg(z)) \qquad \Leftrightarrow \\
\frac{z}{|z|} - \mathrm{i}\,\sin(\arg(z)) &= \sqrt{1-\sin(\arg(z))^2} \qquad \Leftrightarrow \\
\left(\frac{z}{|z|}\right)^2 - 2\,\mathrm{i}\,\frac{z}{|z|}\,\sin(\arg(z)) - \sin(\arg(z))^2 &= 1-\sin(\arg(z))^2 \qquad \Leftrightarrow \\
\sin(\arg(z)) = \frac{\frac{z}{|z|} - \frac{|z|}{z}}{2\,\mathrm{i}} &= \frac{\mathrm{e}^{\mathrm{i}\,\arg(z)} - \mathrm{e}^{-\mathrm{i}\,\arg(z)}}{2\,\mathrm{i}}\,. \qquad (87)
\end{aligned}$$

The results (85) and (87) confirm Pythagoras (84) and the product structure (83). Concerning these results is new, that the logarithm of a number $\frac{z}{|z|}$ of absolute value unity yields an angle and therefore reproduces the arc tangent function:

$$\arg(z) \;=\; \frac{\ln\left(\frac{z}{|z|}\right)}{\mathrm{i}} = \arctan\left(\frac{z-\overline{z}}{z+\overline{z}}\right)\,. \qquad (88)$$

With this turns out, that the logarithm is the very angular function, because by it each angle from 0 until $2\,\pi$ is covered, while the usual arc tangent is only in the range from $-\frac{\pi}{2}$ until $\frac{\pi}{2}$. Of course, this connection is not the case in general, but here it works only, because the angle is given in the correct scaling, namely in the *radial arc* without dimension. This is often clarified at the following limit, which works *only* in the *radial arc* and then gives unity in geometry:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x)}{x} &= \lim_{x\to 0}\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2\,\mathrm{i}\,x} = \lim_{x\to 0}\frac{(1+\mathrm{i}\,x+x^2(\ldots))-(1-\mathrm{i}\,x+x^2(\ldots))}{2\,\mathrm{i}\,x} = \\
&= \lim_{x\to 0}\frac{2\,\mathrm{i}\,x + x^2(\ldots)}{2\,\mathrm{i}\,x} = 1\,. \qquad (89)
\end{aligned}$$

Not at all, this is the only possibility to demonstrate the radial arc as the right *angle scaling*. Rather, the following calculation possibility of number π results:

$$\begin{aligned}
\cos(\pi) &= -1 = \frac{\mathrm{e}^{\mathrm{i}\,\pi}+\mathrm{e}^{-\mathrm{i}\,\pi}}{2} \qquad \Leftrightarrow \\
\mathrm{e}^{2\,\mathrm{i}\,\pi} + 2\,\mathrm{e}^{\mathrm{i}\,\pi} + 1 &= \left(\mathrm{e}^{\mathrm{i}\,\pi}+1\right)^2 = 0 \qquad \Leftrightarrow \\
\mathrm{e}^{\mathrm{i}\,\pi} &= -1 = \mathrm{e}^{-\mathrm{i}\,\pi} \qquad \Leftrightarrow \\
\pi &= \frac{\ln(-1)}{\mathrm{i}}\,. \qquad (90)
\end{aligned}$$

The result (90) *cannot* be found via *Newton's iteration* (32). In general, Newton's iteration fails always, if $f'(x_n)$ is near zero or exact zero. Therefore, alternatives to Newton's iteration (32) are needed. What works very well contrary to this, is the result $\pi = 4\,\arctan(1)$, which can also be found on many pocket calculators, programming languages and so on, for what the following *derivative* is needed:

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} \;=\; \frac{\mathrm{d}\left(\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}\right)}{2\,\mathrm{i}\,\mathrm{d}x} = \frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2} = \cos(x)\,. \qquad (91)$$

This result (91) can also be found in geometry, with more calculation effort via the limit (89). Analogously follows: $\frac{\mathrm{d}\cos(x)}{\mathrm{d}x} = -\sin(x)$.

4.7 Newton's Iterations

4.7.1 Taylor's Series

The series of the *exponential function* (65) has been generalized by *Taylor*, where he found the following connection[53]:

$$f(a) \;=\; \sum_{\mu=0}^{\infty} \left.\frac{\frac{\mathrm{d}^\mu f(a)}{\mathrm{d}a^\mu}}{\mu!}\right|_{a\to x} (a-x)^\mu \,. \tag{92}$$

Because of $\mu!$ in the denominator of (92), this series can be broken off arbitrarily as approximation.

4.7.2 Newton's Iteration of 1^{st} Degree

Now, if *Taylor's series* is broken off after the linear term, then results with the demand $f(a) = 0$, because a zero position is searched for:

$$\begin{aligned} f(a) &= f(x) + f'(x)\,(a-x) \;=\; 0 \qquad \Leftrightarrow \\ a - x &= -\frac{f(x)}{f'(x)} \qquad \Leftrightarrow \\ a &= x - \frac{f(x)}{f'(x)}\,. \end{aligned} \tag{93}$$

For $a \to x$, the Taylor's series (92) becomes exact with: $f(a) = f(x)$. Now in equation (93) is set $x \to x_n$ and $a \to x_{n+1}$, by what the formula (32) is motivated. This iteration method is always unsuitable, if $f'(x) \approx 0$ is valid.

4.7.3 Preparations

For the following example, the *quotient rule* (94) is needed:

$$\frac{\mathrm{d}\left(\frac{f(x)}{g(x)}\right)}{\mathrm{d}x} \;=\; \frac{g(x)\,f'(x) - f(x)\,g'(x)}{g(x)^2}\,, \tag{94}$$

which is a consequence of *product rule* (95):

$$\begin{aligned} \frac{\mathrm{d}\,(f(x)\,g(x))}{\mathrm{d}x} &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x} = \\ &= \lim_{\Delta x\to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\, g(x+\Delta x) \;+\; f(x)\,\frac{g(x+\Delta x) - g(x)}{\Delta x}\right) = \\ &= f'(x)\,g(x) + f(x)\,g'(x) \end{aligned} \tag{95}$$

and *chain rule* (96):

$$\begin{aligned} \frac{\mathrm{d}f(g(x))}{\mathrm{d}x} &= \lim_{\Delta x\to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \cdot \frac{g(x+\Delta x) - g(x)}{g(x+\Delta x) - g(x)} = \\ &= \lim_{\Delta x\to 0} \frac{f(g(x+\Delta x) - g(x) + g(x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} = \\ &= \lim_{\Delta g\to 0} \frac{f(g+\Delta g) - f(g)}{\Delta g} \cdot \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}\,. \end{aligned} \tag{96}$$

[53] [1987BSGZZ], section 3.1.5.3., page 269

4.7.4 Example $\pi = 4\ \arctan(1)$

Now with this follows the function structure of *Newton's iteration of* 1^{st} *degree* for the example $f(x) = \tan\left(\frac{x}{4}\right) - 1$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan\left(\frac{x_n}{4}\right) - 1}{\frac{1}{4\ \cos\left(\frac{x_n}{4}\right)^2}} = x_n - 4\ \cos\left(\frac{x_n}{4}\right)^2 \left(\frac{\sin\left(\frac{x_n}{4}\right)}{\cos\left(\frac{x_n}{4}\right)} - 1\right) =$$

$$= x_n - 2\ \sin\left(\frac{x_n}{2}\right) + 2\ \cos\left(\frac{x_n}{2}\right) + 2\,. \tag{97}$$

Therefore now the iteration yields, starting at $x_0 = 0$:

$x_1 = 4,0000000$
$x_2 = 3,3491115$
$x_3 = 3,1527212$
$x_4 = 3,1416237$
$x_5 = 3,1415927$
$x_6 = 3,1415927$

4.7.5 Example $\pi = 2\ \arccos(0)$

Since $\sin\left(\frac{\pi}{2}\right) = 1$ is valid, here the method (32) of 1^{st} degree can be applied. With $f(x) = \cos\left(\frac{x}{2}\right) - 0$ and $f'(x) = -\frac{1}{2}\ \sin\left(\frac{x}{2}\right)$ results:

$$x_{n+1} = x_n + 2\,\frac{\cos\left(\frac{x_n}{2}\right)}{\sin\left(\frac{x_n}{2}\right)} = x_n + 2\ \cot\left(\frac{x_n}{2}\right)\,. \tag{98}$$

By this follows the iteration from the starting value $x_0 = \pm 1$, because the starting value $x_0 \to 0$ leads to a singularity, by which result 2 solutions $\pm\pi$:

$x_1 = 4,6609754$	$x_1 = -4,6609754$
$x_2 = 2,7612470$	$x_2 = -2,7612470$
$x_3 = 3,1462451$	$x_3 = -3,1462451$
$x_4 = 3,1415926$	$x_4 = -3,1415926$
$x_5 = 3,1415927$	$x_5 = -3,1415927$
$x_6 = 3,1415927$	$x_6 = -3,1415927$

4.7.6 Newton's Iteration of 2^{nd} Degree

Now, Taylor's series (92) is broken off not before the quadratic series term, by which results with $f(a) = 0$, concerning a new search for zero positions:

$$\begin{aligned} f(a) &= f(x) + (a-x)f'(x) + (a-x)^2\frac{f''(x)}{2} = 0 \quad \Leftrightarrow \\ &\frac{f''(x)}{2}\,(a-x)^2 + (a-x)\,f'(x) + f(x) = 0 \quad \Leftrightarrow \\ a - x &= -\frac{f'(x)}{f''(x)} \pm \sqrt{\left(\frac{f'(x)}{f''(x)}\right)^2 - 2\,\frac{f(x)}{f''(x)}} \quad \Rightarrow \\ x_{n+1} &= x_n - \frac{f'(x_n) \mp \sqrt{f'(x_n)^2 - 2\,f(x_n)\,f''(x_n)}}{f''(x_n)}\,. \end{aligned} \tag{99}$$

The *square root* can be calculated via (32). Also here, the zero position $a = x$ is determined exactly with $f(a) = f(x) = 0$. These both methods (99) fail for $f''(x) \approx 0$.

4.7.7 Example $\pi = \frac{\ln(-1)}{\mathrm{i}} = \arccos(-1)$

Since $\sin(\pi) = 0$ is valid, here the method (32) of first degree fails. With $f(x) = \cos(x) + 1$, $f'(x) = -\sin(x)$, and $f''(x) = -\cos(x)$ results by the method (99) of 2^nd^ degree:

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{\sin(x_n)}{\cos(x_n)} \pm \frac{\sqrt{\sin(x_n)^2 + 2\,(\cos(x_n)+1)\,\cos(x_n)}}{\cos(x_n)} = \\
&= x_n - \tan(x_n) \pm \frac{\sqrt{1 + 2\,\cos(x_n) + \cos(x_n)^2}}{\cos(x_n)} = \\
&= x_n - \tan(x_n) \pm \frac{1 + \cos(x_n)}{\cos(x_n)}\,. \qquad (100)
\end{aligned}
$$

Indeed, this method works on two analogous calculation ways.
By this follows the iteration, starting with $x_0 = 0$ and yielding 2 solutions $\pm\pi$:

x_1	$=$	$2,0000000$	x_1	$=$	$-2,0000000$
x_2	$=$	$2,7820419$	x_2	$=$	$-2,7820419$
x_3	$=$	$3,0896186$	x_3	$=$	$-3,0896186$
x_4	$=$	$3,1402873$	x_4	$=$	$-3,1402873$
x_5	$=$	$3,1415918$	x_5	$=$	$-3,1415918$
x_6	$=$	$3,1415927$	x_6	$=$	$-3,1415927$
x_7	$=$	$3,1415927$	x_7	$=$	$-3,1415927$

4.7.8 Example $\ln(2)$

The method (99) yields a further calculation way to calculate $\ln(2)$. $f(x) = \exp(x) - 2$ and $f'(x) = f''(x) = \exp(x)$ is valid for this:

$$
\begin{aligned}
x_{n+1} &= x_n - 1 \pm \frac{\sqrt{\exp(x_n)^2 - 2\,(\exp(x_n) - 2)\,\exp(x_n)}}{\exp(x_n)} = \\
&= x_n - 1 \pm \frac{\sqrt{4\,\exp(x_n) - \exp(x_n)^2}}{\exp(x_n)} = x_n - 1 \pm \sqrt{\frac{4}{\exp(x_n)} - 1}\,. \qquad (101)
\end{aligned}
$$

The iteration yields, starting with $x_0 = 0$, where the version with $-\sqrt{\ldots}$ is divergent:

x_1	$=$	$0,7320508$
x_2	$=$	$0,6931371$
x_3	$=$	$0,6931472$
x_4	$=$	$0,6931472$

Therefore, a third calculation way to calculate $\ln(2)$ has been found.

4.7.9 Newton's Iterations of Higher Degree

In principle, also Newton's iterations of higher degree can be built analogously:

- For 1^st^ degree (32), linear algebra and the derivative rules are sufficient.
- For 2^nd^ degree (99), quadratic algebra and the derivative rules are sufficient.
- For 3^rd^ degree, cubic algebra and the derivative rules are sufficient.
- For n^{th} degree, algebra of n^{th} degree and the derivative rules are sufficient.

Here, finding the algebra of n^{th} degree turns out to be the greater problem.

4.8 Quadratic Functions

4.8.1 Definition

As *quadratic function* is called a function, of which the building of the *inverse function* needs the solution of a *quadratic equation*. For checking of the result, always at least two checks must be calculated, what now is demonstrated.

4.8.2 Hyperbolic Sine

The *hyperbolic sine*[54] is defined the following[55]:

$$y = \sinh(x) \quad := \quad \mathrm{i}\,\sin\left(\frac{x}{\mathrm{i}}\right) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = -\sinh(-x)\,. \tag{102}$$

Its *inverse function* is called *inverse hyperbolic sine*[56] and is built the following[57]:

$$\begin{aligned} x = \operatorname{arsinh}(y) &= \operatorname{arsinh}\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) \qquad \Leftrightarrow \\ 2\,y &= \mathrm{e}^x - \mathrm{e}^{-x} \qquad \Leftrightarrow \\ (\mathrm{e}^x)^2 - 2\,y\,(\mathrm{e}^x) - 1 &= 0 \qquad \Leftrightarrow \\ (\mathrm{e}^x)_{1,2} &= y \pm \sqrt{y^2+1} \qquad \Leftrightarrow \\ x_{1,2} &= \operatorname{arsinh}(y)_{1,2} = \ln\left(y \pm \sqrt{y^2+1}\right)\,. \end{aligned}$$

The checking calculations yield:

$$\begin{aligned} x &= \operatorname{arsinh}(y) = \ln\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \pm \sqrt{\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)^2 + 1}\right) = \ln\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \pm \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\,, \\ y &= \sinh(x) = \frac{y \pm \sqrt{y^2+1} - \frac{1}{y \pm \sqrt{y^2+1}}}{2} = \frac{y \pm \sqrt{y^2+1} + y \mp \sqrt{y^2+1}}{2} = y\,. \end{aligned}$$

Here turns out, that only + leads to the solution[58], and not just ±:

$$x = \operatorname{arsinh}(y) \quad = \quad \ln\left(y + \sqrt{y^2+1}\right)\,. \tag{103}$$

This means, that also the *inverse function* of a *quadratic function* can be *unambiguous*. To check these facts, at least two checking calculations are to be done always. A third check results for example by a diagram of the functions. The inverse hyperbolic sine has got a real solution for $y \geq 0$.

[54] Latin: *sinus hyperbolicus*
[55] [1987BSGZZ], section 2.5.2.3.1., page 187
[56] Latin: *area sinus hyperbolicus*
[57] [1987BSGZZ], section 2.5.2.3.4., page 189
[58] [1987BSGZZ], section 2.5.2.3.4., page 189

4.8.3 Hyperbolic Cosine

The *hyperbolic cosine*[59] is defined the following[60]:

$$y = \cosh(x) \quad := \quad \cos\left(\frac{x}{\mathrm{i}}\right) = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} = \cosh(-x)\,. \tag{104}$$

Its *inverse function* is called *inverse hyperbolic cosine*[61] and is built the following[62]:

$$\begin{aligned} \pm x = \operatorname{arcosh}(y) &= \operatorname{arcosh}\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}\right) \qquad \Leftrightarrow \\ 2\,y &= \mathrm{e}^{x} + \mathrm{e}^{-x} \qquad \Leftrightarrow \\ (\mathrm{e}^{x})^2 - 2\,y\ (\mathrm{e}^{x}) + 1 &= 0 \qquad \Leftrightarrow \\ (\mathrm{e}^{x})_{1,2} &= y \pm \sqrt{y^2 - 1} \qquad \Leftrightarrow \\ x_{1,2} &= \operatorname{arcosh}(y)_{1,2} = \ln\left(y \pm \sqrt{y^2-1}\right)\,. \end{aligned}$$

The checking calculations yield:

$$\begin{aligned} x &= \operatorname{arcosh}(y) = \ln\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} \pm \sqrt{\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}\right)^2 - 1}\right) = \ln\left(\frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} \pm \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}\right)\,, \\ y &= \cosh(\pm x) = \frac{y \pm \sqrt{y^2-1} + \frac{1}{y \pm \sqrt{y^2-1}}}{2} = \frac{y \pm \sqrt{y^2-1} + y \mp \sqrt{y^2-1}}{2} = y\,. \end{aligned}$$

Here turns out, that $\pm$ leads to the solution[63]:

$$x = \operatorname{arcosh}(y) \quad = \quad \ln\left(y \pm \sqrt{y^2-1}\right)\,. \tag{105}$$

This means, that the *inverse function* of a *quadratic function* can be *ambiguous*. To check these facts, at least two checking calculations are to be done always. A third check results for example by a diagram of the functions. The inverse hyperbolic cosine has got a real solution for $y \geq 1$ only.
The following *hyperbolic equation*[64] exists, which justifies the names *hyperbolic sine* and *hyperbolic cosine*:

$$\cosh(x)^2 - \sinh(x)^2 \quad = \quad 1\,. \tag{106}$$

The correctness of these facts (106) results after inserting the definitions (104) and (102) by expanding multiplication, or from geometric considerations.
Furthermore, the following connection is valid:

$$\sinh(x) + \cosh(x) \quad = \quad \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2} = \mathrm{e}^{x}\,. \tag{107}$$

[59] Latin: *cosinus hyperbolicus*
[60] [1987BSGZZ], section 2.5.2.3.1., page 187
[61] Latin: *area cosinus hyperbolicus*
[62] [1987BSGZZ], section 2.5.2.3.4., page 189
[63] [1987BSGZZ], section 2.5.2.3.4., page 189
[64] [1987BSGZZ], sections 2.5.2.3.3. and 2.6.6.1., page 188 and 224

4.8.4 Hyperbolic Tangent

The *hyperbolic tangent*[65] is defined the following[66]:

$$y = \tanh(x) := \mathrm{i} \tan\left(\frac{x}{\mathrm{i}}\right) = \frac{\sinh(x)}{\cosh(x)} = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} = -\tanh(-x)\,. \tag{108}$$

Its *inverse function* is called *hyperbolic arc tangent*[67] and is built the following[68]:

$$\begin{aligned}
x = \operatorname{artanh}(y) &= \operatorname{artanh}\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}\right) \quad \Leftrightarrow \\
\left(\mathrm{e}^x + \mathrm{e}^{-x}\right) y &= \mathrm{e}^x - \mathrm{e}^{-x} \quad \Leftrightarrow \\
(y-1)\,(\mathrm{e}^x)^2 + y + 1 &= 0 \quad \Leftrightarrow \\
(\mathrm{e}^x)_{1,2} &= \pm\sqrt{\frac{1+y}{1-y}} \quad \Leftrightarrow \\
x_{1,2} &= \operatorname{artanh}(y)_{1,2} = \ln\left(\pm\sqrt{\frac{1+y}{1-y}}\right)\,.
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
x &= \operatorname{artanh}(y) = \ln\left(\pm\sqrt{\frac{1+\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}}{1-\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}}}\right) = \ln\left(\pm\sqrt{\frac{2\,\mathrm{e}^x}{2\,\mathrm{e}^{-x}}}\right) = \ln\left(\pm\mathrm{e}^x\right)\,, \\
y &= \tanh(x) = \frac{\pm\sqrt{\frac{1+y}{1-y}} \mp \sqrt{\frac{1-y}{1+y}}}{\pm\sqrt{\frac{1+y}{1-y}} \pm \sqrt{\frac{1-y}{1+y}}} = \frac{1+y-(1-y)}{1+y+1-y} = \frac{2\,y}{2} = y\,.
\end{aligned}$$

Here turns out, that only + leads to the solution[69], and not just ±:

$$x = \operatorname{artanh}(y) = \ln\left(\sqrt{\frac{1+y}{1-y}}\right) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)\,. \tag{109}$$

The hyperbolic arc tangent has got a real solution for $-1 \leq y \leq 1$ only.

4.8.5 Hyperbolic Cotangent

The *hyperbolic cotangent*[70] is defined the following[71]:

$$y = \coth(x) := \frac{\cot\left(\frac{x}{\mathrm{i}}\right)}{\mathrm{i}} = \frac{\cosh(x)}{\sinh(x)} = \frac{1}{\tanh(x)} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = -\coth(-x)\,. \tag{110}$$

[65] Latin: *tangens hyperbolicus*
[66] [1987BSGZZ], section 2.5.2.3.1., page 187
[67] Latin: *area tangens hyperbolicus*
[68] [1987BSGZZ], section 2.5.2.3.4., page 189
[69] [1987BSGZZ], section 2.5.2.3.4., page 189
[70] Latin: *cotangens hyperbolicus*
[71] [1987BSGZZ], section 2.5.2.3.1., page 187

Its *inverse function* is called *inverse hyperbolic cotangent*[72] and is built the following[73]:

$$\begin{aligned}
x = \operatorname{arcoth}(y) &= \operatorname{arcoth}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) \quad \Leftrightarrow \\
\left(e^x - e^{-x}\right) y &= e^x + e^{-x} \quad \Leftrightarrow \\
(y-1)\left(e^x\right)^2 - y - 1 &= 0 \quad \Leftrightarrow \\
\left(e^x\right)_{1,2} &= \pm\sqrt{\frac{y+1}{y-1}} \quad \Leftrightarrow \\
x_{1,2} &= \operatorname{arcoth}(y)_{1,2} = \ln\left(\pm\sqrt{\frac{y+1}{y-1}}\right).
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
x &= \operatorname{arcoth}(y) = \ln\left(\pm\sqrt{\frac{\frac{e^x+e^{-x}}{e^x-e^{-x}}+1}{\frac{e^x+e^{-x}}{e^x-e^{-x}}-1}}\right) = \ln\left(\pm\sqrt{\frac{2\,e^x}{2\,e^{-x}}}\right) = \ln\left(\pm e^x\right), \\
y &= \coth(x) = \frac{\pm\sqrt{\frac{y+1}{y-1}} \pm \sqrt{\frac{y-1}{y+1}}}{\pm\sqrt{\frac{y+1}{y-1}} \mp \sqrt{\frac{y-1}{y+1}}} = \frac{y+1+(y-1)}{y+1-(y-1)} = \frac{2\,y}{2} = y\,.
\end{aligned}$$

Here turns out, that only + leads to the solution[74], and not just ±:

$$x = \operatorname{arcoth}(y) = \ln\left(\sqrt{\frac{y+1}{y-1}}\right) = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right). \tag{111}$$

The inverse hyperbolic cotangent has got a real solution for $-1 \leq \frac{1}{y} \leq 1$ only.
The following identity[75] exists, which can be understood by canceling:

$$\tanh(x)\,\coth(x) = \frac{\sinh(x)}{\cosh(x)}\,\frac{\cosh(x)}{\sinh(x)} = 1\,. \tag{112}$$

4.8.6 Hyperbolic Secant

The *hyperbolic secant*[76] is defined the following[77]:

$$y = \operatorname{sech}(x) := \sec\left(\frac{x}{i}\right) = \frac{\tanh(x)}{\sinh(x)} = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} = \operatorname{sech}(-x)\,. \tag{113}$$

Its *inverse function* is called *inverse hyperbolic secant*[78] and is built the following:

$$\pm x = \operatorname{arsech}(y) = \operatorname{arsech}\left(\frac{2}{e^x + e^{-x}}\right) \quad \Leftrightarrow$$

[72] Latin: *area cotangens hyperbolicus*
[73] [1987BSGZZ], section 2.5.2.3.4., page 189
[74] [1987BSGZZ], section 2.5.2.3.4., page 189
[75] [1987BSGZZ], section 2.5.2.3.3., page 188
[76] Latin: *secans hyperbolicus*
[77] [1987BSGZZ], section 2.5.2.3.1., page 187
[78] Latin: *area secans hyperbolicus*

$$\begin{aligned}
\frac{2}{y} &= \mathrm{e}^x + \mathrm{e}^{-x} \qquad \Leftrightarrow \\
(\mathrm{e}^x)^2 - \frac{2}{y}\,(\mathrm{e}^x) + 1 &= 0 \qquad \Leftrightarrow \\
(\mathrm{e}^x)_{1,2} &= \frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} \qquad \Leftrightarrow \\
x_{1,2} &= \operatorname{arsech}(y)_{1,2} = \ln\left(\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}\right) .
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
x &= \operatorname{arsech}(y) = \ln\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \pm \sqrt{\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)^2 - 1}\right) = \ln\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \pm \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) , \\
y &= \operatorname{sech}(\pm x) = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} + \frac{1}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}}} = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} + \frac{1}{y} \mp \sqrt{\frac{1}{y^2} - 1}} = y .
\end{aligned}$$

Here $\pm$ leads to the solution and can be expressed by (105):

$$x = \operatorname{arsech}(y) = \ln\left(\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}\right) = \operatorname{arcosh}\left(\frac{1}{y}\right) . \tag{114}$$

The inverse hyperbolic secant has got a real solution for $0 \leq y \leq 1$ only.
The following identity[79] exists, which can be understood by expanding multiplication:

$$\operatorname{sech}(x)^2 + \tanh(x)^2 = \frac{1 + \sinh(x)^2}{\cosh(x)^2} = \frac{\cosh(x)^2}{\cosh(x)^2} = 1 . \tag{115}$$

4.8.7 Hyperbolic Cosecant

The *hyperbolic cosecant*[80] is defined the following[81]:

$$y = \operatorname{csch}(x) := \frac{\csc\left(\frac{x}{\mathrm{i}}\right)}{\mathrm{i}} = \frac{\coth(x)}{\cosh(x)} = \frac{1}{\sinh(x)} = \frac{2}{\mathrm{e}^x - \mathrm{e}^{-x}} = -\operatorname{csch}(-x) . \tag{116}$$

Its *inverse function* is called *inverse hyperbolic cosecant*[82] and is built the following:

$$\begin{aligned}
x = \operatorname{arcsch}(y) &= \operatorname{arcsch}\left(\frac{2}{\mathrm{e}^x - \mathrm{e}^{-x}}\right) \qquad \Leftrightarrow \\
\frac{2}{y} &= \mathrm{e}^x - \mathrm{e}^{-x} \qquad \Leftrightarrow \\
(\mathrm{e}^x)^2 - \frac{2}{y}\,(\mathrm{e}^x) - 1 &= 0 \qquad \Leftrightarrow \\
(\mathrm{e}^x)_{1,2} &= \frac{1}{y} \pm \sqrt{\frac{1}{y^2} + 1} \qquad \Leftrightarrow \\
x_{1,2} &= \operatorname{arcsch}(y)_{1,2} = \ln\left(\frac{1}{y} \pm \sqrt{\frac{1}{y^2} + 1}\right) .
\end{aligned}$$

[79] [1987BSGZZ], section 2.5.2.3.3., page 188
[80] Latin: *cosecans hyperbolicus*
[81] [1987BSGZZ], section 2.5.2.3.1., page 187
[82] Latin: *area cosecans hyperbolicus*

The checking calculations yield:

$$x = \operatorname{arcsch}(y) = \ln\left(\frac{e^x - e^{-x}}{2} \pm \sqrt{\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1}\right) = \ln\left(\frac{e^x - e^{-x}}{2} \pm \frac{e^x + e^{-x}}{2}\right),$$

$$y = \operatorname{csch}(x) = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2}+1} - \frac{1}{\frac{1}{y}\pm\sqrt{\frac{1}{y^2}+1}}} = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2}+1} + \frac{1}{y} \mp \sqrt{\frac{1}{y^2}+1}} = y\,.$$

Here, only + leads to the solution and can be expressed by (103):

$$x = \operatorname{arcsch}(y) = \ln\left(\frac{1}{y} + \sqrt{\frac{1}{y^2}+1}\right) = \operatorname{arsinh}\left(\frac{1}{y}\right). \tag{117}$$

The inverse hyperbolic cosecant has got a real solution for $y \geq 0$ only.
The following identity[83] exists, which can be understood by expanding multiplication and (106):

$$\coth(x)^2 - \operatorname{csch}(x)^2 = \frac{\cosh(x)^2 - 1}{\sinh(x)^2} = \frac{\sinh(x)^2}{\sinh(x)^2} = 1\,. \tag{118}$$

4.8.8 Sine

The *sine*[84] is defined the following:

$$y = \sin(x) := \frac{\sinh(\mathrm{i}\,x)}{\mathrm{i}} = \frac{e^{\mathrm{i}\,x} - e^{-\mathrm{i}\,x}}{2\,\mathrm{i}} = -\sin(-x)\,. \tag{119}$$

Its *inverse function* is called *arc sine*[85] and is built the following[86]:

$$\begin{aligned}
x = \arcsin(y) &= \arcsin\left(\frac{e^{\mathrm{i}\,x} - e^{-\mathrm{i}\,x}}{2\,\mathrm{i}}\right) \quad \Leftrightarrow \\
2\,\mathrm{i}\,y &= e^{\mathrm{i}\,x} - e^{-\mathrm{i}\,x} \quad \Leftrightarrow \\
\left(e^{\mathrm{i}\,x}\right)^2 - 2\,\mathrm{i}\,y\left(e^{\mathrm{i}\,x}\right) - 1 &= 0 \quad \Leftrightarrow \\
\left(e^{\mathrm{i}\,x}\right)_{1,2} &= \mathrm{i}\,y \pm \sqrt{1-y^2} \quad \Leftrightarrow \\
x_{1,2} &= \arcsin(y)_{1,2} = \frac{\ln\left(\mathrm{i}\,y \pm \sqrt{1-y^2}\right)}{\mathrm{i}}\,.
\end{aligned}$$

The checking calculations yield:

$$x = \arcsin(y) = \frac{\ln\left(\frac{e^{\mathrm{i}\,x}-e^{-\mathrm{i}\,x}}{2} \pm \sqrt{1 - \left(\frac{e^{\mathrm{i}\,x}-e^{-\mathrm{i}\,x}}{2\,\mathrm{i}}\right)^2}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{e^{\mathrm{i}\,x}-e^{-\mathrm{i}\,x}}{2} \pm \frac{e^{\mathrm{i}\,x}+e^{-\mathrm{i}\,x}}{2}\right)}{\mathrm{i}},$$

$$y = \sin(x) = \frac{\mathrm{i}\,y \pm \sqrt{1-y^2} - \frac{1}{\mathrm{i}\,y \pm \sqrt{1-y^2}}}{2\,\mathrm{i}} = \frac{\mathrm{i}\,y \pm \sqrt{1-y^2} + \mathrm{i}\,y \mp \sqrt{1-y^2}}{2\,\mathrm{i}} = y\,.$$

[83] [1987BSGZZ], section 2.5.2.3.3., page 188
[84] Latin: *sinus* for *arc*
[85] Latin: *arcus sinus* for *radial arc angle*, where the sine of which is x.
[86] [1987BSGZZ], section 2.5.2.1.6., page 184

Here turns out, that only + leads to the solution, and not just ±:

$$x = \arcsin(y) = \frac{\ln\left(\mathrm{i}\,y + \sqrt{1-y^2}\right)}{\mathrm{i}} = \arg\left(\mathrm{i}\,y + \sqrt{1-y^2}\right). \qquad (120)$$

The arc sine has got a real solution for $-1 \leq y \leq 1$ only. Then $\mathrm{i}\,y + \sqrt{1-y^2}$ is a complex number with absolute value $\sqrt{1-y^2+y^2} = 1$, thus positioned on the *unity circle*.

4.8.9 Cosine

The *cosine*[87] is defined the following:

$$y = \cos(x) := \cosh(\mathrm{i}\,x) = \frac{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}}{2} = \cos(-x). \qquad (121)$$

Its *inverse function* is called *arc cosine*[88] and is built the following:

$$\begin{aligned}
\pm x = \arccos(y) &= \arccos\left(\frac{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}}{2}\right) \quad \Leftrightarrow \\
2\,y &= \mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x} \quad \Leftrightarrow \\
\left(\mathrm{e}^{\mathrm{i}\,x}\right)^2 - 2\,y\left(\mathrm{e}^{\mathrm{i}\,x}\right) + 1 &= 0 \quad \Leftrightarrow \\
\left(\mathrm{e}^{\mathrm{i}\,x}\right)_{1,2} &= y \pm \sqrt{y^2-1} \quad \Leftrightarrow \\
x_{1,2} &= \arccos(y)_{1,2} = \frac{\ln\left(y \pm \mathrm{i}\sqrt{1-y^2}\right)}{\mathrm{i}}.
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
\pm x &= \arccos(y) = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2} \pm \sqrt{\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)^2 - 1}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2} \pm \frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)}{\mathrm{i}}, \\
y &= \cos(\pm x) = \frac{y \pm \mathrm{i}\sqrt{1-y^2} + \frac{1}{y \pm \mathrm{i}\sqrt{1-y^2}}}{2} = \frac{y \pm \mathrm{i}\sqrt{1-y^2} + y \mp \mathrm{i}\sqrt{1-y^2}}{2} = y.
\end{aligned}$$

Here turns out, that ± leads to the solution:

$$x = \arccos(y) = \frac{\ln\left(y \pm \mathrm{i}\sqrt{1-y^2}\right)}{\mathrm{i}} = \arg\left(y \pm \mathrm{i}\sqrt{1-y^2}\right). \qquad (122)$$

The arc cosine has got a real solution for $-1 \leq y \leq 1$ only.
The following *circle equation*[89] exists, which justifies the trigonometric names *sine* and *cosine*, agreeing to *Pythagoras' theorem*:

$$\cos(x)^2 + \sin(x)^2 = 1. \qquad (123)$$

The correctness of these facts (123) results after inserting of the definitions (119) and (121) by expanding multiplication, or from geometric considerations.
Furthermore, the following connection is valid:

$$\cos(x) + \mathrm{i}\,\sin(x) = \frac{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}}{2} + \mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}}{2\,\mathrm{i}} = \mathrm{e}^{\mathrm{i}\,x}. \qquad (124)$$

[87]Latin: *cosinus*
[88]Latin: *arcus cosinus*
[89][1987BSGZZ], sections 2.5.2.1.3. and 2.6.6.1., page 180 and 222–223

4.8.10 Tangent

The *tangent*[90] is defined the following[91]:

$$y = \tan(x) \quad := \quad \frac{\tanh(\mathrm{i}\,x)}{\mathrm{i}} = \frac{\sin(x)}{\cos(x)} = \frac{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{i}\,(\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x})} = -\tan(-x)\,. \qquad (125)$$

Its *inverse function* is called *arc tangent*[92] and is built the following:

$$\begin{aligned}
x = \arctan(y) &= \arctan\left(\frac{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{i}\,(\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x})}\right) \quad \Leftrightarrow \\
\left(\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}\right)\mathrm{i}\,y &= \mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x} \quad \Leftrightarrow \\
(\mathrm{i}\,y - 1)\left(\mathrm{e}^{\mathrm{i}\,x}\right)^2 + \mathrm{i}\,y + 1 &= 0 \quad \Leftrightarrow \\
\left(\mathrm{e}^{\mathrm{i}\,x}\right)_{1,2} &= \pm\sqrt{\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}} \quad \Leftrightarrow \\
x_{1,2} &= \arctan(y)_{1,2} = \frac{\ln\left(\pm\sqrt{\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}}\right)}{\mathrm{i}}\,.
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
x &= \arctan(y) = \frac{\ln\left(\pm\sqrt{\frac{1+\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}}{1-\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}}}\right)}{\mathrm{i}} = \frac{\ln\left(\pm\sqrt{\frac{2\,\mathrm{e}^{\mathrm{i}\,x}}{2\,\mathrm{e}^{-\mathrm{i}\,x}}}\right)}{\mathrm{i}} = \frac{\ln\left(\pm\mathrm{e}^{\mathrm{i}\,x}\right)}{\mathrm{i}}\,, \\
y &= \tan(x) = \frac{\pm\sqrt{\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}} \mp \sqrt{\frac{1-\mathrm{i}\,y}{1+\mathrm{i}\,y}}}{\mathrm{i}\left(\pm\sqrt{\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}} \pm \sqrt{\frac{1-\mathrm{i}\,y}{1+\mathrm{i}\,y}}\right)} = \frac{1+\mathrm{i}\,y-(1-\mathrm{i}\,y)}{\mathrm{i}\,(1+\mathrm{i}\,y+1-\mathrm{i}\,y)} = \frac{2\,\mathrm{i}\,y}{2\,\mathrm{i}} = y\,.
\end{aligned}$$

Here turns out, that only + leads to the solution[93], and not just ±:

$$x = \arctan(y) \quad = \quad \frac{\ln\left(\sqrt{\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{1+\mathrm{i}\,y}{\sqrt{1+y^2}}\right)}{\mathrm{i}} = \arg\left(1+\mathrm{i}\,y\right)\,. \qquad (126)$$

The arc tangent has got a real solution for all real y, too.
For comparison, $\arg(x + \mathrm{i}\,y) = \arg(z)$ yields a result in the full rotation, while the *main value* (88) is restricted for real arguments y to results x with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The notion *main value* comes from the fact, that to the logarithm always an integer multiple of $2\,\pi\,\mathrm{i}$ can yet be added, thus all possible results are taken into account. If this integer number is zero, then the result of the logarithms is also called *main value*. As a consequence, all arc functions of trigonometry have got a period of $2\,\pi$, which does not always occur explicitly in the result.
Therefore, some programming languages offer the traditional arc tangent (88) with one real argument and also the expanded arc tangent with two real arguments.

[90]Latin: *tangens*
[91][1987BSGZZ], section 2.5.2.1.3., page 180
[92]Latin: *arc tangent*
[93][1987BSGZZ], section 2.5.2.3.4., page 189

4.8.11 Cotangent

The *cotangent*[94] is defined the following[95]:

$$y = \cot(x) \;:=\; \mathrm{i}\coth(\mathrm{i}\,x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)} = \mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}} = -\cot(-x)\,. \quad (127)$$

Its *inverse function* is called *arc cotangent*[96] and is built the following:

$$\begin{aligned}
x = \operatorname{arccot}(y) &= \operatorname{arccot}\left(\mathrm{i}\,\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}\right) \quad \Leftrightarrow\\
\left(\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}\right)y &= \mathrm{i}\left(\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}\right) \quad \Leftrightarrow\\
(y-\mathrm{i})\left(\mathrm{e}^{\mathrm{i}\,x}\right)^2 - y - \mathrm{i} &= 0 \quad \Leftrightarrow\\
\left(\mathrm{e}^{\mathrm{i}\,x}\right)_{1,2} &= \pm\sqrt{\frac{y+\mathrm{i}}{y-\mathrm{i}}} \quad \Leftrightarrow\\
x_{1,2} &= \operatorname{arccot}(y)_{1,2} = \frac{\ln\left(\pm\sqrt{\frac{y+\mathrm{i}}{y-\mathrm{i}}}\right)}{\mathrm{i}}\,.
\end{aligned}$$

The checking calculations yield:

$$\begin{aligned}
x &= \operatorname{arccot}(y) = \frac{\ln\left(\pm\sqrt{\frac{\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}+\mathrm{i}}{\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}-\mathrm{i}}}\right)}{\mathrm{i}} = \frac{\ln\left(\pm\sqrt{\frac{2\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x}}{2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,x}}}\right)}{\mathrm{i}} = \frac{\ln\left(\pm\mathrm{e}^{\mathrm{i}\,x}\right)}{\mathrm{i}}\,,\\
y &= \cot(x) = \mathrm{i}\,\frac{\pm\sqrt{\frac{y+\mathrm{i}}{y-\mathrm{i}}}\pm\sqrt{\frac{y-\mathrm{i}}{y+\mathrm{i}}}}{\pm\sqrt{\frac{y+\mathrm{i}}{y-\mathrm{i}}}\mp\sqrt{\frac{y-\mathrm{i}}{y+\mathrm{i}}}} = \mathrm{i}\,\frac{y+\mathrm{i}+(y-\mathrm{i})}{y+\mathrm{i}-(y-\mathrm{i})} = \frac{2\,\mathrm{i}\,y}{2\,\mathrm{i}} = y\,.
\end{aligned}$$

Here turns out, that only + leads to the solution, and not just ±:

$$x = \operatorname{arccot}(y) = \frac{\ln\left(\sqrt{\frac{y+\mathrm{i}}{y-\mathrm{i}}}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{y+\mathrm{i}}{\sqrt{y^2+1}}\right)}{\mathrm{i}} = \arctan\left(\frac{1}{y}\right) = \arg\left(y+\mathrm{i}\right)\,. \quad (128)$$

The arc cotangent has got a real solution for all real y. Also the arc cotangent can be programmed as a function with two real arguments, usually the following:

$$\operatorname{arccot}(y,x) = \arctan(x,y) = \arg(x+\mathrm{i}\,y) = \frac{\ln\left(\frac{x+\mathrm{i}\,y}{\sqrt{x^2+y^2}}\right)}{\mathrm{i}}\,. \quad (129)$$

The following identity[97] exists, which can be understood by canceling:

$$\tan(x)\,\cot(x) = \frac{\sin(x)}{\cos(x)}\,\frac{\cos(x)}{\sin(x)} = 1\,. \quad (130)$$

[94] Latin: *cotangens*
[95] [1987BSGZZ], section 2.5.2.1.3., page 180
[96] Latin: *arcus cotangens*
[97] [1987BSGZZ], section 2.5.2.1.3., page 180

4.8.12 Secant

The *secant*[98] is defined the following[99]:

$$y = \sec(x) \quad := \quad \operatorname{sech}(\mathrm{i}\,x) = \frac{\tan(x)}{\sin(x)} = \frac{1}{\cos(x)} = \frac{2}{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}} = \sec(-x)\,. \tag{131}$$

Its *inverse function* is called *arc secant*[100] and is built the following:

$$\begin{aligned} \pm x = \operatorname{arcsec}(y) &= \operatorname{arcsec}\left(\frac{2}{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}}\right) \quad \Leftrightarrow \\ \frac{2}{y} &= \mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x} \quad \Leftrightarrow \\ \left(\mathrm{e}^{\mathrm{i}\,x}\right)^2 - \frac{2}{y}\left(\mathrm{e}^{\mathrm{i}\,x}\right) + 1 &= 0 \quad \Leftrightarrow \\ \left(\mathrm{e}^{\mathrm{i}\,x}\right)_{1,2} &= \frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} \quad \Leftrightarrow \\ x_{1,2} &= \operatorname{arcsec}(y)_{1,2} = \frac{\ln\left(\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}\right)}{\mathrm{i}}\,. \end{aligned}$$

The checking calculations yield:

$$\begin{aligned} \pm x &= \operatorname{arcsec}(y) = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2} \pm \sqrt{\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)^2 - 1}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2} \pm \frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)}{\mathrm{i}}\,, \\ y &= \sec(\pm x) = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} + \frac{1}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}}} = \frac{2}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1} + \frac{1}{y} \mp \sqrt{\frac{1}{y^2} - 1}} = y\,. \end{aligned}$$

Here, $\pm$ leads to the solution and can be expressed by (122):

$$x = \operatorname{arcsec}(y) = \frac{\ln\left(\frac{1}{y} \pm \mathrm{i}\sqrt{1 - \frac{1}{y^2}}\right)}{\mathrm{i}} = \arccos\left(\frac{1}{y}\right)\,. \tag{132}$$

The arc secant has got a real solution for $-1 \leq \frac{1}{y} \leq 1$ only.
The following identity[101] exists, which can be understood by (123):

$$\sec(x)^2 - \tan(x)^2 = \frac{1 - \sin(x)^2}{\cos(x)^2} = \frac{\cos(x)^2}{\cos(x)^2} = 1\,. \tag{133}$$

4.8.13 Cosecant

The *cosecant*[102] is defined the following[103]:

$$y = \csc(x) \quad := \quad \mathrm{i}\operatorname{csch}(\mathrm{i}\,x) = \frac{\cot(x)}{\cos(x)} = \frac{1}{\sin(x)} = \frac{2\,\mathrm{i}}{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}} = -\csc(-x)\,. \tag{134}$$

[98] Latin: *secans*
[99] [1987BSGZZ], section 2.5.2.1.1., page 178
[100] Latin: *arcus secans*
[101] [1987BSGZZ], section 2.5.2.1.3., page 180
[102] Latin: *cosecans*
[103] [1987BSGZZ], section 2.5.2.1.1., page 178

Its *inverse function* is called *arc cosecant*[104] and is built the following:

$$\begin{aligned} x \;=\; \operatorname{arccsc}(y) &= \operatorname{arccsc}\left(\frac{2\,\mathrm{i}}{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}\right) \qquad \Leftrightarrow \\ \frac{2\,\mathrm{i}}{y} &= \mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x} \qquad \Leftrightarrow \\ \left(\mathrm{e}^{\mathrm{i}\,x}\right)^2-\frac{2\,\mathrm{i}}{y}\left(\mathrm{e}^{\mathrm{i}\,x}\right)-1 &= 0 \qquad \Leftrightarrow \\ \left(\mathrm{e}^{\mathrm{i}\,x}\right)_{1,2} &= \frac{\mathrm{i}}{y}\pm\sqrt{1-\frac{1}{y^2}} \qquad \Leftrightarrow \\ x_{1,2} &= \operatorname{arccsc}(y)_{1,2} = \frac{\ln\left(\frac{\mathrm{i}}{y}\pm\sqrt{1-\frac{1}{y^2}}\right)}{\mathrm{i}}\,. \end{aligned}$$

The checking calculations yield:

$$\begin{aligned} x &= \operatorname{arccsc}(y) = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2}\pm\sqrt{1+\left(\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)^2}\right)}{\mathrm{i}} = \frac{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2}\pm\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)}{\mathrm{i}}\,, \\ y &= \csc(x) = \frac{2\,\mathrm{i}}{\frac{\mathrm{i}}{y}\pm\sqrt{1-\frac{1}{y^2}}-\frac{1}{\frac{\mathrm{i}}{y}\pm\sqrt{1-\frac{1}{y^2}}}} = \frac{2\,\mathrm{i}}{\frac{\mathrm{i}}{y}\pm\sqrt{1-\frac{1}{y^2}}+\frac{\mathrm{i}}{y}\mp\sqrt{1-\frac{1}{y^2}}} = y\,. \end{aligned}$$

Here, only + leads to the solution and can be expressed by (120):

$$x = \operatorname{arccsc}(y) = \frac{\ln\left(\frac{\mathrm{i}}{y}+\sqrt{1-\frac{1}{y^2}}\right)}{\mathrm{i}} = \arcsin\left(\frac{1}{y}\right). \tag{135}$$

The arc cosecant has got a real solution for $-1 \le \frac{1}{y} \le 1$ only.
The following identity[105] exists, which can be understood by (123):

$$\csc(x)^2-\cot(x)^2 = \frac{1-\cos(x)^2}{\sin(x)^2} = \frac{\sin(x)^2}{\sin(x)^2} = 1\,. \tag{136}$$

4.9 Derivatives of Quadratic Functions

4.9.1 Derivative of the Hyperbolic Sine

The derivative of the *hyperbolic sine* (102) yields[106]:

$$\frac{\mathrm{d}\sinh(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}\right)}{\mathrm{d}x} = \frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2} = \cosh(x)\,. \tag{137}$$

The derivative of the *inverse hyperbolic sine* (103) yields[107]:

$$\frac{\mathrm{d}\operatorname{arsinh}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\ln\left(y+\sqrt{y^2+1}\right)\right)}{\mathrm{d}y} = \frac{1+\frac{2\,y}{2\sqrt{y^2+1}}}{y+\sqrt{y^2+1}} = \frac{1}{\sqrt{y^2+1}}\,. \tag{138}$$

[104] Latin: *arcus cosecans*
[105] [1987BSGZZ], section 2.5.2.1.3., page 180
[106] [1987BSGZZ], section 1.1.3.3., integral number 427., page 60
[107] [1987BSGZZ], section 1.1.3.3., integral number 192., page 46

4.9.2 Derivative of the Hyperbolic Cosine

The derivative of the *hyperbolic cosine* (104) yields[108]:

$$\frac{\mathrm{d}\cosh(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}\right)}{\mathrm{d}x} = \frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2} = \sinh(x)\,. \tag{139}$$

The derivative of the *inverse hyperbolic cosine* (105) yields[109]:

$$\frac{\mathrm{d}\,\mathrm{arcosh}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\ln\left(y\pm\sqrt{y^2-1}\right)\right)}{\mathrm{d}y} = \frac{1\pm\frac{2\,y}{2\sqrt{y^2-1}}}{y\pm\sqrt{y^2-1}} = \frac{\pm1}{\sqrt{y^2-1}}\,. \tag{140}$$

4.9.3 Derivative of the Hyperbolic Tangent

The derivative of the *hyperbolic tangent* (108) yields[110]:

$$\frac{\mathrm{d}\tanh(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\sinh(x)}{\cosh(x)}\right)}{\mathrm{d}x} = \frac{\cosh(x)^2-\sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2}\,. \tag{141}$$

The derivative of the *hyperbolic arc tangent* (109) yields[111]:

$$\frac{\mathrm{d}\,\mathrm{artanh}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)\right)}{\mathrm{d}y} = \frac{1}{2}\,\frac{1-y}{1+y}\,\frac{(1-y)+(1+y)}{(1-y)^2} = \frac{1}{1-y^2}\,. \tag{142}$$

4.9.4 Derivative of the Hyperbolic Cotangent

The derivative of the *hyperbolic cotangent* (110) yields[112]:

$$\frac{\mathrm{d}\coth(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\cosh(x)}{\sinh(x)}\right)}{\mathrm{d}x} = \frac{\sinh(x)^2-\cosh(x)^2}{\sinh(x)^2} = \frac{-1}{\sinh(x)^2}\,. \tag{143}$$

The derivative of the *inverse hyperbolic cotangent* (111) yields[113]:

$$\frac{\mathrm{d}\,\mathrm{arcoth}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)\right)}{\mathrm{d}y} = \frac{1}{2}\,\frac{y-1}{y+1}\,\frac{(y-1)-(y+1)}{(y-1)^2} = \frac{1}{1-y^2}\,. \tag{144}$$

Since the derivatives (142) and (144) are the same, $\mathrm{artanh}(y)$ and $\mathrm{arcoth}(y)$ distinguish by a constant only, where the *main value* of which can be determined for $y=0$ the easiest:

$$\mathrm{arcoth}(y)-\mathrm{artanh}(y) = \mathrm{arcoth}(0)-\mathrm{artanh}(0) = \frac{\ln(-1)-\ln(1)}{2} = \frac{\mathrm{i}\,\pi}{2}\,. \tag{145}$$

This result is analogous to the sum (160) of $\arctan(y)$ and $\mathrm{arccot}(y)$[114].

[108][1987BSGZZ], section 1.1.3.3., integral number 426., page 60
[109][1987BSGZZ], section 1.1.3.3., integral number 220., page 48
[110][1987BSGZZ], section 1.1.3.3., integral number 431., page 60
[111][1987BSGZZ], section 1.1.3.3., integral number 57., page 38
[112][1987BSGZZ], section 1.1.3.3., integral number 430., page 60
[113][1987BSGZZ], section 1.1.3.3., integral number 57., page 38
[114][1987BSGZZ], section 2.5.2.1.7., page 185

4.9.5 Derivative of the Hyperbolic Secant

The derivative of the *hyperbolic secant* (113) yields:

$$\frac{\mathrm{d}\,\mathrm{sech}(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{\cosh(x)}\right)}{\mathrm{d}x} = \frac{\cosh(x)\,0 - 1\,\sinh(x)}{\cosh(x)^2} = -\mathrm{sech}(x)\,\tanh(x)\,. \qquad (146)$$

The derivative of the *inverse hyperbolic secant* (114) yields[115]:

$$\frac{\mathrm{d}\,\mathrm{arsech}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\ln\left(\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}\right)\right)}{\mathrm{d}y} = \frac{-\frac{1}{y^2} \pm \frac{-\frac{2}{y^3}}{2\sqrt{\frac{1}{y^2}-1}}}{\frac{1}{y} \pm \sqrt{\frac{1}{y^2} - 1}} = \frac{\mp 1}{y\,\sqrt{1-y^2}}\,. \qquad (147)$$

An independent calculation way results by (140):

$$\frac{\mathrm{d}\,\mathrm{arsech}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\,\mathrm{arcosh}\left(\frac{1}{y}\right)}{\mathrm{d}y} = \frac{\pm 1}{\sqrt{\frac{1}{y^2} - 1}}\left(\frac{-1}{y^2}\right) = \frac{\mp 1}{y\,\sqrt{1-y^2}}\,.$$

4.9.6 Derivative of the Hyperbolic Cosecant

The derivative of the *hyperbolic cosecant* (116) yields:

$$\frac{\mathrm{d}\,\mathrm{csch}(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{\sinh(x)}\right)}{\mathrm{d}x} = \frac{\sinh(x)\,0 - 1\,\cosh(x)}{\sinh(x)^2} = -\mathrm{csch}(x)\,\coth(x)\,. \qquad (148)$$

The derivative of the *inverse hyperbolic cosecant* (117) yields[116]:

$$\frac{\mathrm{d}\,\mathrm{arcsch}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\ln\left(\frac{1}{y} + \sqrt{\frac{1}{y^2} + 1}\right)\right)}{\mathrm{d}y} = \frac{-\frac{1}{y^2} + \frac{-\frac{2}{y^3}}{2\sqrt{\frac{1}{y^2}+1}}}{\frac{1}{y} + \sqrt{\frac{1}{y^2} + 1}} = \frac{-1}{y\,\sqrt{1+y^2}}\,. \qquad (149)$$

An independent calculation way results by (138):

$$\frac{\mathrm{d}\,\mathrm{arcsch}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\,\mathrm{arsinh}\left(\frac{1}{y}\right)}{\mathrm{d}y} = \frac{1}{\sqrt{\frac{1}{y^2} + 1}}\left(\frac{-1}{y^2}\right) = \frac{1}{y\,\sqrt{1+y^2}}\,.$$

4.9.7 Derivative of the Sine

The derivative of the *sine* (119) yields[117]:

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}}{2\,\mathrm{i}}\right)}{\mathrm{d}x} = \frac{\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}}{2} = \cos(x)\,. \qquad (150)$$

[115] [1987BSGZZ], section 1.1.3.3., integral number 168., page 45
[116] [1987BSGZZ], section 1.1.3.3., integral number 196., page 46
[117] [1987BSGZZ], section 1.1.3.3., integral number 313., page 54

The derivative of the *arc sine* (120) yields[118]:

$$\frac{\mathrm{d}\,\arcsin(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\,\ln\left(\mathrm{i}\,y+\sqrt{1-y^2}\right)\right)}{\mathrm{d}y} = \frac{\mathrm{i}-\frac{2\,y}{2\sqrt{1-y^2}}}{\left(\mathrm{i}\,y+\sqrt{1-y^2}\right)\,\mathrm{i}} = $$

$$= \frac{\mathrm{i}\sqrt{1-y^2}-y}{-y+\mathrm{i}\sqrt{1-y^2}}\cdot\frac{1}{\sqrt{1-y^2}} = \frac{1}{\sqrt{1-y^2}}\,. \qquad (151)$$

The derivatives of the inverse functions of quadratic functions motivate to think on *integrating reduction of fractions* to higher terms, when searching for *base integrals*.

4.9.8 Derivative of the Cosine

The derivative of the *cosine* (121) yields[119]:

$$\frac{\mathrm{d}\cos(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{e}^{\mathrm{i}\,x}+\mathrm{e}^{-\mathrm{i}\,x}}{2}\right)}{\mathrm{d}x} = \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,x}-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\,x}}{2} = \mathrm{i}^2\left(\frac{\mathrm{e}^{\mathrm{i}\,x}-\mathrm{e}^{-\mathrm{i}\,x}}{2\,\mathrm{i}}\right) = -\sin(x)\,. \qquad (152)$$

The derivative of the *arc cosine* (122) yields:

$$\frac{\mathrm{d}\arccos(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\,\ln\left(y\pm\mathrm{i}\sqrt{1-y^2}\right)\right)}{\mathrm{d}y} = \frac{1\mp\mathrm{i}\,\frac{2\,y}{2\sqrt{1-y^2}}}{\left(y\pm\mathrm{i}\sqrt{1-y^2}\right)\,\mathrm{i}} = \frac{\mp 1}{\sqrt{1-y^2}}\,. \qquad (153)$$

An independent calculation way results due to the derivation of (122):

$$\frac{\mathrm{d}\arccos(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\,\ln\left(y\pm\sqrt{y^2-1}\right)\right)}{\mathrm{d}y} = \frac{1\pm\frac{2\,y}{2\sqrt{y^2-1}}}{\left(y\pm\sqrt{y^2-1}\right)\,\mathrm{i}} = \frac{\pm 1}{\mathrm{i}\sqrt{y^2-1}} = \frac{\mp 1}{\sqrt{1-y^2}}\,.$$

Here, the comparison of this result with the result (153) yields:

$$\sqrt{y^2-1} = \mathrm{i}\sqrt{1-y^2} \qquad \Leftrightarrow \qquad \mathrm{i}\sqrt{y^2-1} = -\sqrt{1-y^2}\,. \qquad (154)$$

This result (154) can help to clear up many sign problems. Indeed, $\mathrm{i}=+\sqrt{-1}$ is *no* positive constant and therefore needs *specific* calculation rules.
The sum or difference of the derivatives (151) and (153) yields zero, thus *arc sine* and *arc cosine* distinguish eventually by a constant[120] only:

$$\arcsin(y)\pm\arccos(y) = \frac{\ln\left(\mathrm{i}\,y+\sqrt{1-y^2}\right)}{\mathrm{i}} \pm \frac{\ln\left(y\pm\mathrm{i}\sqrt{1-y^2}\right)}{\mathrm{i}} = \frac{\ln(\mathrm{i})}{\mathrm{i}} = \frac{\pi}{2}\,. \qquad (155)$$

An independent calculaton way results due to the derivation of (122) with (154):

$$\arcsin(y)\pm\arccos(y) = \frac{\ln\left(\mathrm{i}\,y+\sqrt{1-y^2}\right)}{\mathrm{i}} \pm \frac{\ln\left(y\pm\sqrt{y^2-1}\right)}{\mathrm{i}} = \frac{\ln(\mathrm{i})}{\mathrm{i}} = \frac{\pi}{2}\,.$$

The result (154) urges to be careful, when placing $\mathrm{i}=+\sqrt{-1}$ outside the brackets.

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[118] [1987BSGZZ], section 1.1.3.3., integral number 164., page 44
[119] [1987BSGZZ], section 1.1.3.3., integral number 274., page 52
[120] [1987BSGZZ], section 2.5.2.1.7., page 185

4.9.9 Derivative of the Tangent

The derivative of the *tangent* (125) yields[121]:

$$\frac{\mathrm{d}\tan(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\sin(x)}{\cos(x)}\right)}{\mathrm{d}x} = \frac{\cos(x)^2+\sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2}\,. \tag{156}$$

The derivative of the *arc tangent* (126) yields[122]:

$$\frac{\mathrm{d}\arctan(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{2\,\mathrm{i}}\ln\left(\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}\right)\right)}{\mathrm{d}y} = \frac{1}{2\,\mathrm{i}}\,\frac{1-\mathrm{i}\,y}{1+\mathrm{i}\,y}\cdot\frac{(1-\mathrm{i}\,y)\,\mathrm{i}+(1+\mathrm{i}\,y)\,\mathrm{i}}{(1-\mathrm{i}\,y)^2} = \frac{1}{1+y^2}\,. \tag{157}$$

4.9.10 Derivative of the Cotangent

The derivative of the *cotangent* (127) yields[123]:

$$\frac{\mathrm{d}\cot(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\cos(x)}{\sin(x)}\right)}{\mathrm{d}x} = \frac{-\sin(x)^2-\cos(x)^2}{\sin(x)^2} = \frac{-1}{\sin(x)^2}\,. \tag{158}$$

The derivative of the *arc cotangent* (128) yields[124]:

$$\frac{\mathrm{d}\,\mathrm{arccot}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{2\,\mathrm{i}}\ln\left(\frac{y+\mathrm{i}}{y-\mathrm{i}}\right)\right)}{\mathrm{d}y} = \frac{1}{2\,\mathrm{i}}\,\frac{y-\mathrm{i}}{y+\mathrm{i}}\,\frac{(y-\mathrm{i})-(y+\mathrm{i})}{(y-\mathrm{i})^2} = \frac{-1}{1+y^2}\,. \tag{159}$$

Since the sum of the derivatives (157) and (159) is zero, the sum of $\arctan(y)$ and $\mathrm{arccot}(y)$ yields a constant, where the *main value*[125] of which enables the calculation of π for all y:

$$\arctan(y)+\mathrm{arccot}(y) = \frac{1}{2\,\mathrm{i}}\ln\left(\frac{1+\mathrm{i}\,y}{1-\mathrm{i}\,y}\cdot\frac{y+\mathrm{i}}{y-\mathrm{i}}\right) = \frac{\ln\left(\frac{\mathrm{i}+\mathrm{i}\,y^2}{-\mathrm{i}-\mathrm{i}\,y^2}\right)}{2\,\mathrm{i}} = \frac{\ln(-1)}{2\,\mathrm{i}} = \frac{\pi}{2}\,. \tag{160}$$

This result is analogous to the difference (145) of $\mathrm{arcoth}(y)$ and $\mathrm{artanh}(y)$.

4.9.11 Derivative of the Secant

The derivative of the *secant* (131) yields[126]:

$$\frac{\mathrm{d}\sec(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{\cos(x)}\right)}{\mathrm{d}x} = \frac{\cos(x)\,0+1\,\sin(x)}{\cos(x)^2} = \sec(x)\,\tan(x)\,. \tag{161}$$

The derivative of the *arc secant* (132) yields[127]:

$$\frac{\mathrm{d}\,\mathrm{arcsec}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\ln\left(\frac{1}{y}\pm\mathrm{i}\sqrt{1-\frac{1}{y^2}}\right)\right)}{\mathrm{d}y} = \frac{-\frac{1}{y^2}\pm\mathrm{i}\,\frac{\frac{2}{y^3}}{2\sqrt{1-\frac{1}{y^2}}}}{\frac{\mathrm{i}}{y}\mp\sqrt{1-\frac{1}{y^2}}} = \frac{\pm 1}{y\,\sqrt{y^2-1}}\,. \tag{162}$$

[121] [1987BSGZZ], section 1.1.3.3., integral number 326., page 54
[122] [1987BSGZZ], section 1.1.3.3., integral number 57., page 38
[123] [1987BSGZZ], section 1.1.3.3., integral number 287., page 52
[124] [1987BSGZZ], section 1.1.3.3., integral number 57., page 38
[125] [1987BSGZZ], section 2.5.2.1.7., page 185
[126] [1987BSGZZ], section 1.1.3.3., integral number 370., page 57
[127] [1987BSGZZ], section 1.1.3.3., integral number 224., page 48

Independent calculation ways result via (153) or by the notation of (132):

$$\frac{\mathrm{d}\,\mathrm{arcsec}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\arccos\left(\frac{1}{y}\right)}{\mathrm{d}y} = \frac{\mp 1}{\sqrt{1-\frac{1}{y^2}}}\cdot\frac{-1}{y^2} = \frac{\pm 1}{y\sqrt{y^2-1}}\,.$$

$$\frac{\mathrm{d}\,\mathrm{arcsec}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\,\ln\left(\frac{1}{y}\pm\sqrt{\frac{1}{y^2}-1}\right)\right)}{\mathrm{d}y} = \frac{-\frac{1}{y^2}\pm\frac{-\frac{2}{y^3}}{2\sqrt{\frac{1}{y^2}-1}}}{\left(\frac{1}{y}\pm\sqrt{\frac{1}{y^2}-1}\right)\,\mathrm{i}} = \frac{\pm 1}{y\sqrt{y^2-1}}\,.$$

These results show in comparison with (154), that for placing of $\mathrm{i} = +\sqrt{-1}$ outside the brackets is *always* to be considered a *context*, which reads *here*:

$$\sqrt{\frac{1}{y^2}-1} = \mathrm{i}\sqrt{1-\frac{1}{y^2}} \quad\Leftrightarrow\quad \mathrm{i}\sqrt{\frac{1}{y^2}-1} = -\sqrt{1-\frac{1}{y^2}}\,. \tag{163}$$

The difference of (154) and (163) is often overlooked for automated, numerical evaluation!

4.9.12 Derivative of the Cosecant

The derivative of the *cosecant* (134) yields[128]:

$$\frac{\mathrm{d}\csc(x)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{1}{\sin(x)}\right)}{\mathrm{d}x} = \frac{\sin(x)\,0 - 1\,\cos(x)}{\sin(x)^2} = -\csc(x)\,\cot(x)\,. \tag{164}$$

The derivative of the *arc cosecant* (135) yields:

$$\frac{\mathrm{d}\,\mathrm{arccsc}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\left(\frac{1}{\mathrm{i}}\,\ln\left(\frac{\mathrm{i}}{y}+\sqrt{1-\frac{1}{y^2}}\right)\right)}{\mathrm{d}y} = \frac{-\frac{\mathrm{i}}{y^2}+\frac{\frac{2}{y^3}}{2\sqrt{1-\frac{1}{y^2}}}}{\left(\frac{\mathrm{i}}{y}+\sqrt{1-\frac{1}{y^2}}\right)\,\mathrm{i}} = \frac{-1}{y\sqrt{y^2-1}}\,. \tag{165}$$

An independent calculation way results by (151):

$$\frac{\mathrm{d}\,\mathrm{arccsc}(y)}{\mathrm{d}y} = \frac{\mathrm{d}\arcsin\left(\frac{1}{y}\right)}{\mathrm{d}y} = \frac{1}{\sqrt{1-\frac{1}{y^2}}}\cdot\frac{-1}{y^2} = \frac{-1}{y\sqrt{y^2-1}}\,.$$

The sum or difference of (162) and (165) yields zero, thus the analogous operation of arc secant (132) and arc cosecant (135) yields a constant:

$$\pm\,\mathrm{arcsec}(y) + \mathrm{arccsc}(y) = \pm\frac{\ln\left(\frac{1}{y}\pm\mathrm{i}\sqrt{1-\frac{1}{y^2}}\right)1}{\mathrm{i}} + \frac{\ln\left(\frac{\mathrm{i}}{y}+\sqrt{1-\frac{1}{y^2}}\right)}{\mathrm{i}} = \frac{\pi}{2}\,. \tag{166}$$

An independent calculation way results via the derivation of (132) with (163):

$$\pm\,\mathrm{arcsec}(y) + \mathrm{arccsc}(y) = \pm\frac{\ln\left(\frac{1}{y}\pm\sqrt{\frac{1}{y^2}-1}\right)1}{\mathrm{i}} + \frac{\ln\left(\frac{\mathrm{i}}{y}+\sqrt{1-\frac{1}{y^2}}\right)}{\mathrm{i}} = \frac{\pi}{2}\,.$$

[128] [1987BSGZZ], section 1.1.3.3., integral number 381., page 57

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