How many Notes has got the Gamut?

Norbert Südland*
Otto–Schott–Straße 16
D–73431 Aalen /Württemberg

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Abstract

Again and again in music theory a lot of things are taken to be unalterable without having thought in detail about their contentional reasons. Examples to these basic assumptions are a gamut of 7 notes or a fifth circle of 12 elements. By some calculation expense and the necessary physical understanding can be shown, that these assumptions in no case are forcing.

1 Notions

From the physical considerations about signal precision the following definitions result:

- *Tone*: A tone is a noise whose oscillation frequency can be determined.
- *Harmony*: Starting with the discussion of the harmonic oscillator integer ratios of oscillation number are called harmony because of their signal precision.
- *Consonance*:
  - If the oscillation number ratio is rational, the tones melt to a unique tone having a characteristic sound. This effect of *perfect consonance* is used to build organ records.
  - If the oscillation number ratio is *somewhat* rational, the result is a beat about the arithmetic mean of the frequencies. By this it is possible to detect still several tones by hearing. This effect is extremely interesting for music for several voices.
- *Dissonance*: The beat frequency is so high, that it disturbs the detection of the basic frequency. The brain is overburden on classifying the frequencies.

The notions given here orientate on the subjective impression and furthermore have got a physical context, on which is based the functionality of electronic tuning boxes.

Especially important is the insight, that harmonic music has got to do with consonances, which depending on the based theory can be performed differently.

*E–Mail: Norbert.Suedland@t-online.de

2 Consequences from Weber's and Fechner's Law

Due to Weber and Fechner nerves respond logarithmically to stimuli, thus a double signal always will be feeled to be the same amplification. In music a double frequency also always is feeled to be the same interval. The double frequency (oscillation number ratio 2:1) in European music tradition is called *octave*. Because of Weber's and Fechner's law a symmetrical partition of the octave is to be orientated on equal frequency ratios and not on equal frequency differences.

The mathematical style is given by the logarithm of the corresponding frequency ratio, where meanwhile in European music tradition has been introduced the unit *Cent* to determine interval distances objectively:

$$1\,\mathrm{Cent} \;:=\; 2^{\frac{1}{1200}} \tag{1}$$

100 Cent are calculated the following:

$$100\,\mathrm{Cent} \;=\; (1\,\mathrm{Cent})^{100} \;=\; 2^{\frac{1}{12}} \tag{2}$$

Even this is a European half–tone of the equalized beating tune. An octave therefore has got 1200 Cent. A given frequency ratio $\frac{f_2}{f_1}$ is calculated to Cent the following:

$$1200\,\frac{\log\left(\frac{f_2}{f_1}\right)}{\log(2)}\,\mathrm{Cent} \tag{3}$$

Since all logarithms are proportional to each other, the calculation rule (3) depends only on using the same logarithm both in the numerator and in the denominator.

In general an "even" Cent value does not show the consonance of an interval at all. This unit will be used here anyway, because it is common.

3 Harmonic Sounds

When looking for *consonance* the rational oscillation number ratios play the important role. Thus results for the intervals from which a harmonic double sound is builded up:

$$\frac{4}{3}\,\frac{3}{2} \;=\; \frac{2}{1} \;=\; \frac{3}{2}\,\frac{4}{3} \tag{4}$$

For the triad yields:

$$\frac{6}{5}\,\frac{5}{4}\,\frac{4}{3} \;=\; \frac{2}{1} \;=\; \frac{4}{3}\,\frac{5}{4}\,\frac{6}{5} \tag{5}$$

For a quadruple sound yields:

$$\frac{8}{7}\,\frac{7}{6}\,\frac{6}{5}\,\frac{5}{4} \;=\; \frac{2}{1} \;=\; \frac{5}{4}\,\frac{6}{5}\,\frac{7}{6}\,\frac{8}{7} \tag{6}$$

For a pentuble sound follows analogeously:

$$\frac{10}{9}\,\frac{9}{8}\,\frac{8}{7}\,\frac{7}{6}\,\frac{6}{5} \;=\; \frac{2}{1} \;=\; \frac{6}{5}\,\frac{7}{6}\,\frac{8}{7}\,\frac{9}{8}\,\frac{10}{9} \tag{7}$$

For the quadruple and pentuble sounds the permutation possibilities are much more than shown. A separation into major and minor anyway results by mirror symmetry also already with the triad.

4 A Note System to Kids

If all, that a child acts on a music instrument, shall sound harmonic, it has to be tuned in a pure pentuble sound[1], and then only harmonic music can result if it is played. The most known piece of music concerning five tune music eventually is called “flea waltz”, which is played on the black keys of a piano only.

If a musical instrument is tuned by a computer, there is a possibility to switch from “adult’s tuning” to “children’s tuning” if required, where yet enough variation possibilities result: Pentuble sound instruments can be tuned purely harmonical in several cases[2].

5 Equally Beating Gamut Intervals

Not to get a specific sound character for each key, it seems to be sensible to choose the distances between two neighboured notes to be constant. By this e.g. a song can be striked up some higher or deeper without tuning the instruments to another pitch[3].

The denotation *equally beating tune* comes from the fact, that for tuning the fifths and fourth are used to be the purest intervals to generate all notes from this. Since no pure intervals[4] are used, the fifths and fourths each own a little beat[5]. Violonists most tune their instrument in pure fifths, what should be taken into consideration during the composition.

The mathematical description of the equally beating tune is:

$$\text{ConcertPitch} \times 2^{\frac{m}{n}} \quad m \in \{\ldots, -1, 0, 1, \ldots\} \quad n \in \{2, 3, \ldots, \} \tag{8}$$

The actual question, how many notes n has got the gamut, however is not answered at all by this. This question can be answered by a longer calculation, where a compromise of consonant intervals is looked for at an equally beating tune.

It is worth to be mentioned, that J. S. Bach gave to his *“Best Tempered Piano”* also a corresponding *Best Tempered Tune*[6], which diverges from the equally beating tune a little bit. This shows, that he was able to deal with the provisionality of his tuning systems.

To be able to judge the quality of the sound of the resulting intervals, the distance to a neighbouring consonance interval is calculated in Cent:

$$\Delta := 1200 \left(\frac{m}{n} - \frac{\log\left(\frac{f_2}{f_1}\right)}{\log(2)} \right) \text{Cent} \tag{9}$$

The evaluation of the calculation leads to preferable separations n of the octave, if the basic intervals are hit possibly well.

[1]By permutating of the intervals there are $5 \times 4! = 5 \times 24$ several possibilities to do this!

[2]namely in 24 several tune classes, each of them having five inversions!

[3]For this however the players must be able to rethink, namely to “transpose”.

[4]except for the octave

[5]in Europe some 2 beats per second

[6]see [Bil1989], p. 30f

6 Results

Useful results yield for $n \in \{5, 7, 12, 22, 41, \ldots\}$. Except for the last mentioned possibility all separations are historically practised—without therefore being also equally beating.

6.1 The Equally Beating 5–Separation

The interval differences Δ are so large, that the harmonic variants (7) compete with it:

m	$2^{m/5}$	$f_2 : f_1$	$\frac{\Delta}{[\text{Cent}]}$
0	1.0000	1 : 1	0.0000
1	1.1487	8 : 7	8.8259
2	1.3195	4 : 3	−18.0450
3	1.5157	3 : 2	18.0450
4	1.7411	7 : 4	−8.8259
5	2.0000	2 : 1	0.0000

In pure tuning result the following symmetrical interval sequences for a pentuble sound, which are neither major nor minor:

$$\frac{8}{7}\frac{7}{6}\frac{9}{8}\frac{7}{6}\frac{8}{7} = \frac{2}{1} = \frac{7}{6}\frac{8}{7}\frac{9}{8}\frac{8}{7}\frac{7}{6} \tag{10}$$

6.2 The Equally Beating 7–Separation

The interval differences Δ are so large, that historically a diatonic gamut resulted having an asymmetric separation:

m	$2^{m/7}$	$f_2{:}f_1$	$\frac{\Delta}{[\text{Cent}]}$
0	1.0000	1:1	0.0000
1	1.1041	11:10	6.4243
2	1.2190	6:5	27.2159
3	1.3459	4:3	16.2407
4	1.4860	3:2	−16.2407
5	1.6407	5:3	−27.2159
6	1.8114	9:5	10.9751
7	2.0000	2:1	0.0000

For $m = 1$ and $m = 6$ the dissonances are so large, that also concerning the calculation it is hard to give an unequivocal interval to them.

A pure tuned symmetric 7–sound looks somewhat like this and shows, that also alternatives to the diatonic[7] gamuts exist:

$$\frac{11}{10}\frac{12}{11}\frac{10}{9}\frac{9}{8}\frac{10}{9}\frac{12}{11}\frac{11}{10} = \frac{2}{1} = \frac{12}{11}\frac{11}{10}\frac{10}{9}\frac{9}{8}\frac{10}{9}\frac{11}{10}\frac{12}{11} \tag{11}$$

[7] consisting of 5 whole tone and 2 half tone intervals of the 12–separated octave

6.3 The Equally Beating 12–Separation

At this separation the equally beating tune has been used for the first time in history:

m	$2^{m/12}$	f_2:f_1	$\frac{\Delta}{[\text{Cent}]}$
0	1.0000	1:1	0.0000
1	1.0595	18:17	1.0454
2	1.1225	9:8	−3.9100
3	1.1892	6:5	−15.6413
4	1.2599	5:4	13.6863
5	1.3348	4:3	1.9550
6	1.4142	7:5	17.4878
7	1.4983	3:2	−1.9550
8	1.5874	8:5	−13.6863
9	1.6818	5:3	15.6413
10	1.7818	16:9	3.9100
11	1.8877	17:9	−1.0454
12	2.0000	2:1	0.0000

6.4 The Equally Beating 22–Separation

This separation is historically known as being non–equally beating from Arabia or India:

m	$2^{m/22}$	f_2:f_1	$\frac{\Delta}{[\text{Cent}]}$	m	$2^{m/22}$	f_2:f_1	$\frac{\Delta}{[\text{Cent}]}$
0	1.0000	1:1	0.0000	22	2.0000	2:1	0.0000
1	1.0320	32:31	−0.4190	21	1.9380	31:16	0.4190
2	1.0650	16:15	−2.6404	20	1.8779	15:8	2.6404
3	1.0991	11:10	−1.3679	19	1.8196	11:6	−12.9993
4	1.1343	8:7	−12.9923	18	1.7632	7:4	12.9923
5	1.1706	7:6	5.8564	17	1.7085	12:7	−5.8564
6	1.2081	6:5	11.6314	16	1.6555	5:3	−11.6314
7	1.2468	5:4	−4.4955	15	1.6042	8:5	4.4955
8	1.2867	9:7	1.2795	14	1.5544	14:9	−1.2795
9	1.3278	4:3	−7.1359	13	1.5062	3:2	7.1359
10	1.3704	11:8	−5.8634	12	1.4595	19:13	−2.4399
11	1.4142	7:5	17.4878	11	1.4142	7:5	17.4878

This separation with equally beating separation owns comparatively bad tuned fifths and fourths, but especially many consonances for melody leading.

6.5 The Equally Beating 41–Separation

This separation is equally beating *and* allows harmonic quartuble (6) and eventually also pentuble (10) sounds. Until now it has not yet been used:

m	$2^{m/41}$	f_2:f_1	$\frac{\Delta}{[\text{Cent}]}$	m	$2^{m/41}$	f_2:f_1	$\frac{\Delta}{[\text{Cent}]}$
0	1.0000	1:1	0.0000	41	2.0000	2:1	0.0000
1	1.0170	60:59	0.1712	40	1.9665	59:30	−0.1712
2	1.0344	30:29	−0.1549	39	1.9335	29:15	0.1549
3	1.0520	20:19	−0.9958	38	1.9011	19:10	0.9958
4	1.0700	15:14	−2.3696	37	1.8692	15:8	−5.3419
5	1.0882	12:11	−4.2956	36	1.8379	11:6	4.2956
6	1.1068	10:9	−6.7940	35	1.8071	9:5	6.7940
7	1.1256	9:8	0.9680	34	1.7768	16:9	−0.9680
8	1.1448	8:7	2.9722	33	1.7470	7:4	−2.9722
9	1.1643	7:6	−3.4563	32	1.7177	12:7	3.4563
10	1.1842	13:11	3.4732	31	1.6889	17:10	−11.3246
11	1.2044	6:5	6.3099	30	1.6606	5:3	−6.3099
12	1.2249	11:9	3.8116	29	1.6328	18:11	−3.8116
13	1.2458	5:4	−5.8259	28	1.6054	8:5	5.8259
14	1.2670	19:15	0.5118	27	1.5785	11:7	7.7519
15	1.2886	9:7	3.9403	26	1.5520	14:9	−3.9403
16	1.3106	21:16	−2.4882	25	1.5260	29:19	−0.3569
17	1.3330	4:3	−0.4840	24	1.5004	3:2	0.4840
18	1.3557	19:14	−1.8578	23	1.4753	31:21	−1.0839
19	1.3788	11:8	4.7796	22	1.4505	16:11	−4.7796
20	1.4023	7:5	2.8537	21	1.4262	10:7	−2.8537

Besides fantastic consonances this separation owns also dissonances and leads to the expectation of a corresponding rich music. Even two different *tritoni*[8] are available, also there is now the ratio 9 : 8 besides 8 : 7 for the *large second.*

The computed notes can be put together to a melody also with a QBASIC program and the command SOUND, while for several voices there is a need of more programming expence. The notes of the 41–separation have yet got a distance so far, that the difference of 29.2683 Cent of two neighboured melody notes can be heared well. With a 200–separation with 6 Cent interval distance this is scarcely possible.

It remains to be questionable, whether it can be expected to an intrumental musician to tune and play a separation of seventh notes of the 41–separation. For this yet a note line system has to be developed to contain the aspired sound volume.

[8]singular: *tritonus*; stands for three *whole notes* of the 7–separation.

7 What is New with this Examination?

The following summary is just a trial to list the extremely unusual aspects of this study:

- The claim, art be something which cannot be discussed objectively, in the case of music is not true.
- The claim, in musics has been all, that is possible with harmonic music, already done, does not show expertise.
- The *pythagoraic comma* by the 12– and 41–separation is fulfilled in best possibility. The glorifying of number 12 in music has been relativated by this.
- The importance of Indian music (22–separation of the octave) can be shown especially for multi–voicing music, while by this the predominance of European music has come to an end.
- The 22–separation and especially the 41–separation of the octave own the advantages of the *"mitteltönigen"* tune, but avoid the problem of the *organ wolf*[9].
- With available compositions can be tried, how much they are transponable to the 41–separation of the octave.
- It is remarkable, that just J. S. Bach was a great lover of number 41 [Sme1950][10]. His compositions, that always was tuned to the preliminarity of the intonation, exspect for the first to get the transposition to the 41–separated octave as an optimizing of sound.
- There is an exspectation, that all tune systems of the world can be summarized by the 41–separation, thus a harmonic mixture of European and oriental music eventually will prosper.
- Well–done music theory is a compromise of good sound and practicability.

The theoretical considerations now must be confirmed by praxis.

References

[Bil1989] Billeter, Bernhard: *Anweisung zum Stimmen von Tasteninstrumenten*, Merseburger, 3rd edition, (1989)

[Sme1950] Smend, Friedrich: *Johann Sebastian Bach bei seinem Namen gerufen*, Bärenreiter, (1950)

[9]The interval as–es by the "mitteltönigen" tune is a fifth, that howls horribly—like a wolf.

[10]p. 29