Hydrostatics.nb

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■ 1.1. Problem

Which pressure is found inside a star?

■ 1.2. Preparation

The original *Mathematica* commands are used.

These commands allow to generate a unit check of the results:

$$\mathbf{UnitCheck} = \left\{\mathrm{Rule} \to \mathrm{Equal},\ p'[\_] \to \frac{\text{"N"}}{\text{"m"}^3},\ g[\_] \to \frac{\text{"m"}}{\text{"s"}^2},\ p[\_] \to \frac{\text{"N"}}{\text{"m"}^2},\ \rho[\_] \to \frac{\text{"kg"}}{\text{"m"}^3},\ g \to \frac{\text{"m"}}{\text{"s"}^2},\ \gamma \to \frac{\text{"m"}^3}{\text{"kg" "s"}^2},\ \rho \to \frac{\text{"kg"}}{\text{"m"}^3},\ h \to \text{"m"},\ H \to \text{"m"},\ R \to \frac{\text{"J"}}{\text{"K" "mol"}},\ T \to \text{"K"},\ M \to \frac{\text{"kg"}}{\text{"mol"}},\ p \to \frac{\text{"N"}}{\text{"m"}^2},\ a \to \frac{\text{"N" "m"}^6}{\text{"m"}^2},\ b \to \text{"m"}^3,\ V \to \text{"m"}^3,\ \text{"N"} \to \frac{\text{"kg" "m"}}{\text{"s"}^2},\ \text{"J"} \to \text{"N" "m"},\ \_?\mathrm{NumericQ}\ \mathrm{term}\_ :\to \mathrm{term}\right\}$$

$$\left\{\mathrm{Rule} \to \mathrm{Equal},\ p'[\_] \to \frac{\mathrm{N}}{\mathrm{m}^3},\ g[\_] \to \frac{\mathrm{m}}{\mathrm{s}^2},\ p[\_] \to \frac{\mathrm{N}}{\mathrm{m}^2},\ \rho[\_] \to \frac{\mathrm{kg}}{\mathrm{m}^3},\ g \to \frac{\mathrm{m}}{\mathrm{s}^2},\ \gamma \to \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2},\ \rho \to \frac{\mathrm{kg}}{\mathrm{m}^3},\ h \to \mathrm{m},\ H \to \mathrm{m},\ R \to \frac{\mathrm{J}}{\mathrm{K\,mol}},\ T \to \mathrm{K},\ M \to \frac{\mathrm{kg}}{\mathrm{mol}},\ p \to \frac{\mathrm{N}}{\mathrm{m}^2},\ a \to \mathrm{m}^4\,\mathrm{N},\ b \to \mathrm{m}^3,\ V \to \mathrm{m}^3,\ \mathrm{N} \to \frac{\mathrm{kg\,m}}{\mathrm{s}^2},\ \mathrm{J} \to \mathrm{m\,N},\ \mathrm{term}\_\_?\mathrm{NumericQ} :\to \mathrm{term}\right\}$$

The change of the default scripture font for plots of any kind is done the following:

$DefaultFont = {"Times", 12.}

{Times, 12.}

1.3. Solution

1.3.1. Hydrostatic Pressure

1.3.1.1. Textbook Formula

Due to the dtv lexicon of physics ([dtv1969], volume **4**, keyword *Hydrostatik [hydrostatics]*, page 160-162) for not too big differences of height $\boldsymbol{h}$ and constant fall acceleration $\boldsymbol{g}$ is valid:

$$p_2 - p_1 == \rho\, g\, h \tag{1.1}$$

1.3.1.2. Pressure Gradient

For $\boldsymbol{\rho \to \rho[h]}$ and $\boldsymbol{g \to g[h]}$ first results the necessity to determine a pressure gradient:

$$\nabla p == \lim_{\Delta h \to 0} \frac{p[h + \Delta h] - p[h]}{\Delta h} == \partial_h\, p[h] == \rho[h]\, g[h] \tag{1.2}$$

1.3.1.3. General Hydrostatic Pressure

The general hydrostatic pressure now results as an integral of this pressure gradient:

$$p[h] == \int \partial_h\, p[h]\, dh == \int \rho[h]\, g[h]\, dh \tag{1.3}$$

1.3.2. Incompressible Liquid

1.3.2.1. Constant Density

For an incompressible liquid $\boldsymbol{\rho[h] \to \rho}$ is a constant, e.g. for an earth that would consists of water only.

1.3.2.2. Fall Acceleration under Water

Then the fall acceleration under water in each case is the fall acceleration that would be generated by the remaining inner sphere due to Newtons's gravity law (see [BeS1945], § 31, page 111-112: There the derivation is not completely valid; see also [Krau2007], figure 9, page 27 and chapter 4, page 32-50), i.e.:

$$\left\{g[h] \to -\frac{\gamma\, 4\,\pi}{h^2} \int_0^h h^2\,\rho\, dh\right\}$$

% //. UnitCheck

$$\left\{g[h] \to -\frac{4}{3}\, h\,\pi\,\gamma\,\rho\right\}$$

{True}

Thus the fall acceleration decreases continuously in the direction to the middle of the liquid star.

■ 1.3.2.3. Hydrostatic Pressure

Now by this the hydrostatic pressure results as an integral of $\rho\, g[h]$:

$$\left\{p[h] \to \int_H^h \rho\, g[h]\, dh\ /.\ \left\{g[h] \to -\frac{4}{3}\, h\,\pi\,\gamma\,\rho\right\}\right\}$$

% //. UnitCheck

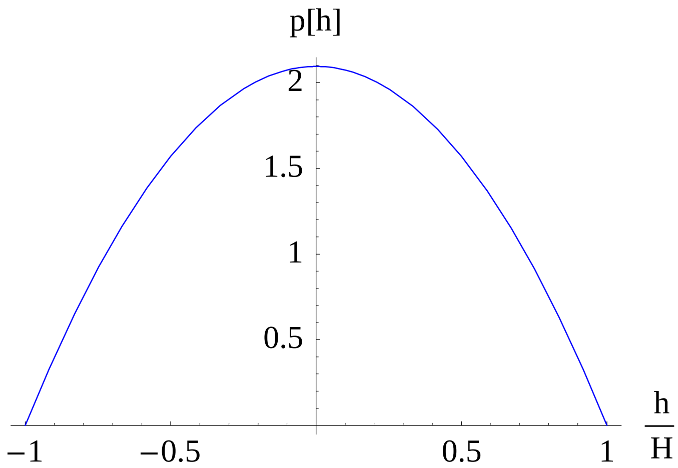
$$\left\{p[h] \to \rho\left(-\frac{2}{3}\, h^2\,\pi\,\gamma\,\rho + \frac{2}{3}\, H^2\,\pi\,\gamma\,\rho\right)\right\}$$

{True}

■ 1.3.2.4. Figure $p[h]$ over h

This yields the following figure:

$$\text{Plot @@ }\left\{\text{Evaluate}\left[\rho\left(-\frac{2}{3}\, h^2\,\pi\,\gamma\,\rho + \frac{2}{3}\,\pi\, H^2\,\gamma\,\rho\right) /.\ \{\gamma \to 1,\ \rho \to 1,\ H \to 1\}\right],\right.$$
$$\left.\{h, -1, 1\},\ \text{AxesLabel} \to \left\{"\frac{h}{H}",\ "p[h]"\right\},\ \text{PlotStyle} \to \text{Hue}\left[\frac{2}{3}\right]\right\};$$



■ 1.3.2.5. Peak Pressure in the Center

The peak pressure in the center is:

$$\rho\left(-\frac{2}{3}h^2\pi\gamma\rho+\frac{2}{3}\pi H^2\gamma\rho\right) /. \{h\to 0\}$$

$$\frac{2}{3}H^2\pi\gamma\rho^2$$

Thus the peak pressure in the center can be determined.

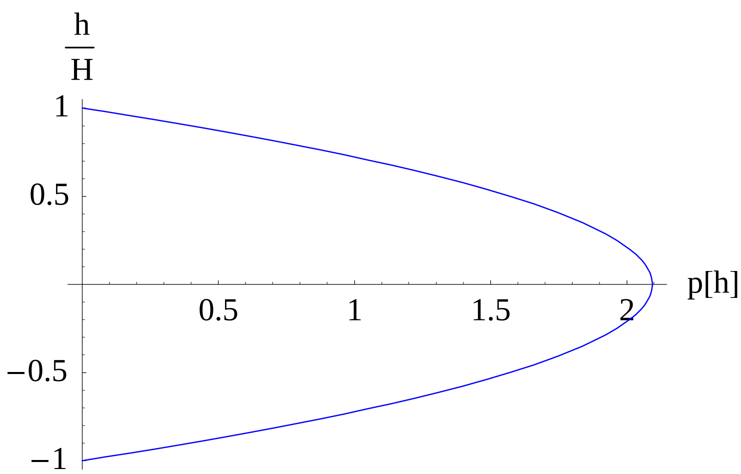
■ 1.3.2.6. Figure *h* over *p*[*h*]

The representation of the inverse function also is sensible:

$$h \;/.\; \text{Solve}\left[p == \rho\left(-\frac{2}{3}h^2\pi\gamma\rho+\frac{2}{3}\pi H^2\gamma\rho\right), h\right] \;//\; \text{Simplify}$$

$$\text{Plot} @@ \left\{\text{Evaluate}[\% \;/.\; \{H\to 1, \gamma\to 1, \rho\to 1\}], \left\{p, 0, \frac{2}{3}\pi H^2\gamma\rho^2 \;/.\; \{H\to 1, \gamma\to 1, \rho\to 1\}\right\}, \text{AxesLabel}\to\left\{\text{"p[h]"}, \text{"}\frac{h}{H}\text{"}\right\}, \text{PlotStyle}\to\text{Hue}\left[\frac{2}{3}\right]\right\};$$

$$\left\{-\sqrt{H^2-\frac{3p}{2\pi\gamma\rho^2}}, \sqrt{H^2-\frac{3p}{2\pi\gamma\rho^2}}\right\}$$



■ 1.3.2.7. Approximation for Low Depth

For not too large depth **Δh** yields the following approximation:

$$\rho\left(-\frac{2}{3}h^2\pi\gamma\rho+\frac{2}{3}\pi H^2\gamma\rho\right)\text{ /. }\{h\to H-\Delta h\}\text{ // Expand}$$

Series[%, {Δh, 0, 1}] // Normal

$$\frac{\%\, g}{\frac{4}{3}H\pi\gamma\rho}$$

$$\frac{4}{3}H\pi\gamma\Delta h\,\rho^2-\frac{2}{3}\pi\gamma\Delta h^2\rho^2$$

$$\frac{4}{3}H\pi\gamma\Delta h\,\rho^2$$

$$g\,\Delta h\,\rho$$

This is the textbook formula of the hydrostatic pressure in an incompressible liquid. Here, the sign of ***g*** was choosen to be positive, because the pressure increases by the depth **Δh**.

■ 1.3.3. Compressible Gas

■ 1.3.3.1. Ideal Gas Law

Now the density is not constant, but itself a function of the pressure, in the easiest case due to the ideal gas law:

$$p\,V == n\,R\,T\text{ /. }\left\{n\to\frac{m}{M}\right\}\text{ /. }\{m\to\rho\,V\}$$

Solve[%, ρ] // Flatten

% //. UnitCheck

$$p\,V == \frac{R\,T\,V\,\rho}{M}$$

$$\left\{\rho\to\frac{M\,p}{R\,T}\right\}$$

{True}

■ 1.3.3.2. Fall Acceleration in a Gas

The fall acceleration in a gas results again in each case as the fall acceleration ***g*[*h*]** that would be generated by the remaining inner sphere due to Newton's gravity law at position ***h***, i.e.:

$$\left\{g[h]\to-\frac{\gamma\,4\,\pi}{h^2}\int_0^h\rho[h]\,h^2\,dh\right\}$$

$$\left\{g[h]\to-\frac{4\,\pi\,\gamma\int_0^h h^2\,\rho[h]\,dh}{h^2}\right\}$$

Thus the fall acceleration decreases continuously and non-linarly with the direction to the center of the gasic star.

1.3.3.3. Non-linear Differential Equation

Now by this the following connection results:

$$\mathbf{PressureEquation}[p] = \partial_h p[h] == \rho[h]\, g[h] \text{ /. } \left\{g[h] \to -\frac{\gamma\, 4\, \pi}{h^2} \int_0^h \rho[h]\, h^2\, dh\right\} \text{ /. } \left\{\rho[h] \to \frac{M\, p[h]}{R\, T}\right\}$$

$$p'[h] == -\frac{4\, M^2\, \pi\, \gamma \left(\int_0^h h^2\, p[h]\, dh\right) p[h]}{h^2\, R^2\, T^2}$$

This is a non-linear integrodifferential equation that can be changed to a pure differential equation after separation of the integral an applying the derivative:

$$\frac{h^2\, \#}{p[h]} \text{ \& /@ PressureEquation}[p]$$

$$\partial_h \# \text{ \& /@ \%}$$

$$\text{PressureEquation}[p, 1] = \frac{\#\, p[h]}{h} \text{ \& /@ \% // ExpandAll}$$

$$\frac{h^2\, p'[h]}{p[h]} == -\frac{4\, M^2\, \pi\, \gamma \int_0^h h^2\, p[h]\, dh}{R^2\, T^2}$$

$$\frac{2\, h\, p'[h]}{p[h]} - \frac{h^2\, p'[h]^2}{p[h]^2} + \frac{h^2\, p''[h]}{p[h]} == -\frac{4\, h^2\, M^2\, \pi\, \gamma\, p[h]}{R^2\, T^2}$$

$$2\, p'[h] - \frac{h\, p'[h]^2}{p[h]} + h\, p''[h] == -\frac{4\, h\, M^2\, \pi\, \gamma\, p[h]^2}{R^2\, T^2}$$

This is a non-linear differential equation of 2nd order in $p[h]$ that cannot be solved easily in an analytic way.

1.3.3.4. Trivial Solution

The solution $p[h] \equiv 0$ fulfills the equation:

$p[h]$ # & /@ PressureEquation[p, 1] // ExpandAll
% /. {p → Function[{h}, 0]}

$$2\, p[h]\, p'[h] - h\, p'[h]^2 + h\, p[h]\, p''[h] == -\frac{4\, h\, M^2\, \pi\, \gamma\, p[h]^3}{R^2\, T^2}$$

True

Thus the vacuum owns the hydrostatic stable solution $p[h] == 0$ for any h. Of course, this is valid only for the areas where no star can be found.

1.3.3.5. Essential Property

The derivated equation yields with $h \to 0$:

```
PressureEquation[p, 1]
% /. {h → 0}
```

$$2\,p'[h]-\frac{h\,p'[h]^2}{p[h]}+h\,p''[h]==-\frac{4\,h\,M^2\,\pi\,\gamma\,p[h]^2}{R^2\,T^2}$$

$$2\,p'[0]==0$$

Thus results **p'[0] → 0**. By this only one free choosable initial value problem remains, e.g. **p[0] → 1**. Deviations from **p'[0] → 0** must lead to singularities in **p[h]**.

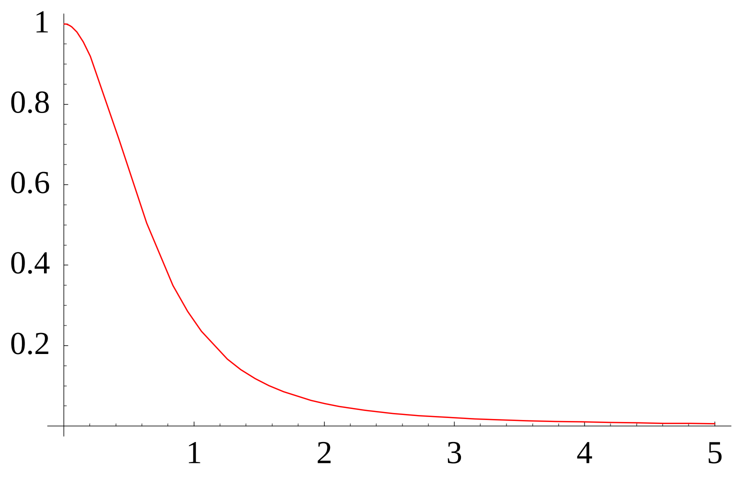
This property **p'[0] → 0** results from the fact, that a radial symmetry has bees assumed for the star, and that in the symmetry center the resulting gravity force is zero, i.e. no further pressure increase. Therefore the trip into the singularities for **p'[0] ≠ 0** can be omitted.

▪ 1.3.3.6. Numerical Solutions and Figures *p*[*h*] over *h*

The numerical solution of this differential equation successes well, if both initial value problems start at the same position **h ≈ 0**:

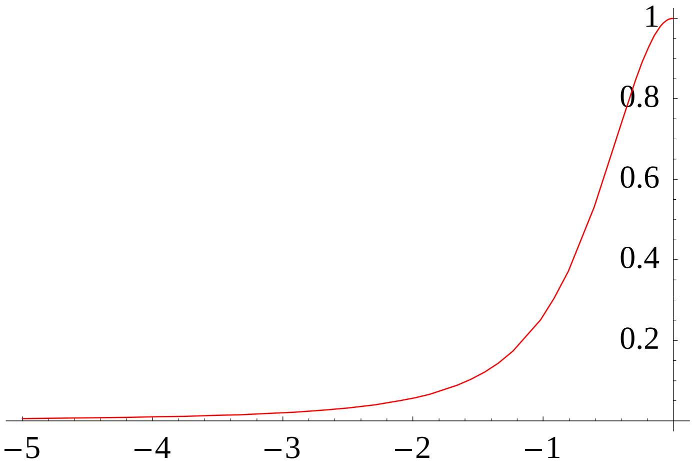
```
Solve[PressureEquation[p, 1], p''[h]] /. {Rule → Equal} // Flatten // Simplify
Curve[RightHandSide, 1] = NDSolve[
    {%[[1]] /. {M → 1, γ → 1, R → 1, T → 1}, p[1/1000] == 1, p'[1/1000] == 0}, p[h], {h, 1/1000, 5}];
Figure[RightHandSide, 1] = Plot @@
    {Evaluate[p[h] /. Curve[RightHandSide, 1]], {h, 1/1000, 5}, PlotStyle → Hue[0]};
```

$$\left\{p''[h]==-\frac{4\,M^2\,\pi\,\gamma\,p[h]^2}{R^2\,T^2}-\frac{2\,p'[h]}{h}+\frac{p'[h]^2}{p[h]}\right\}$$



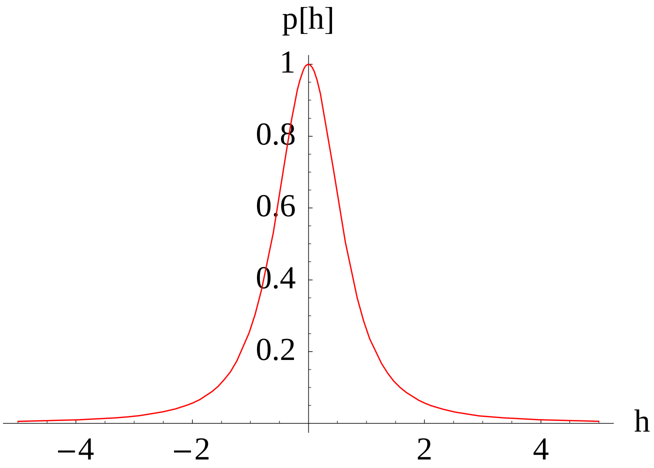
```
Solve[PressureEquation[p, 1], p''[h]] /. {Rule → Equal} // Flatten // Simplify
Curve[LeftHandSide, 1] =
    NDSolve[{%[[1]] /. {M → 1, γ → 1, R → 1, T → 1}, p[-1/1000] == 1, p'[-1/1000] == 0},
     p[h], {h, -1/1000, -5}];
Figure[LeftHandSide, 1] = Plot @@ {Evaluate[p[h] /. Curve[LeftHandSide, 1]],
       {h, -1/1000, -5}, PlotStyle → Hue[0]};
```

$$\left\{p''[h] == -\frac{4 M^2 \pi \gamma p[h]^2}{R^2 T^2} - \frac{2 p'[h]}{h} + \frac{p'[h]^2}{p[h]}\right\}$$



The superposition of both curves is very convincing:

```
Figure[1] = Show[Figure[LeftHandSide, 1], Figure[RightHandSide, 1], AxesLabel → {"h", "p[h]"}];
```



For another parameter choice a similar figure is found:

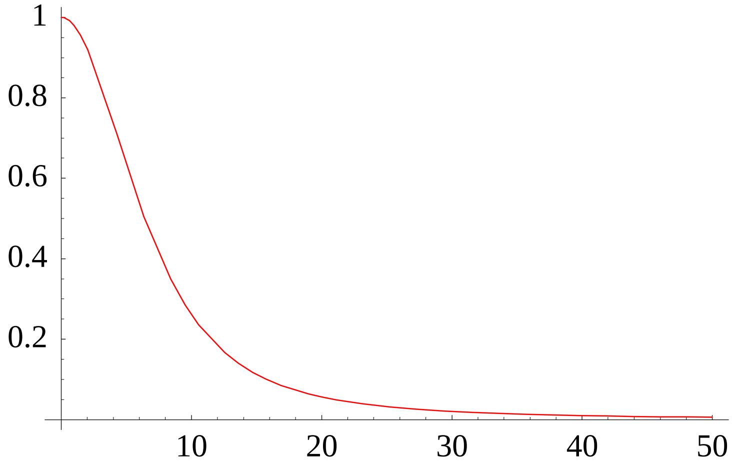
Solve[PressureEquation[*p*, 1], *p*''[*h*]] /. {Rule → Equal} // Flatten // Simplify
Curve[RightHandSide, 2] = NDSolve[

$$\left\{\%[[1]] \,/.\, \{M \to 1, \gamma \to 1, R \to 1, T \to 10\},\ p\left[\frac{1}{1000}\right] == 1,\ p'\left[\frac{1}{1000}\right] == 0\right\},\ p[h],\ \left\{h, \frac{1}{1000}, 50\right\}\right];$$

Figure[RightHandSide, 2] = Plot @@ {Evaluate[*p*[*h*] /. Curve[RightHandSide, 2]],

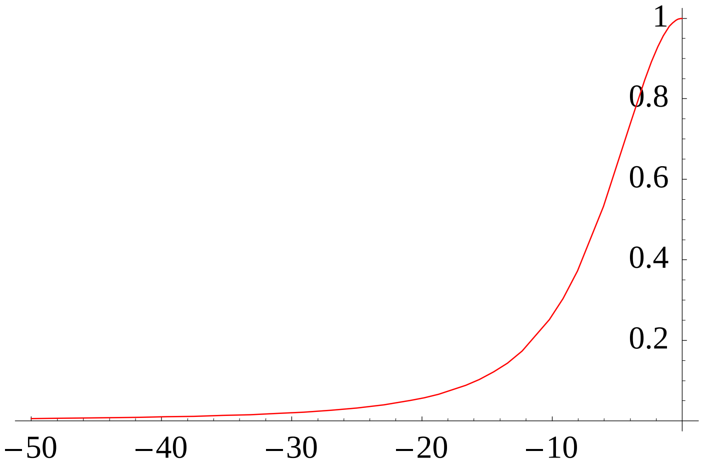
$$\left\{h, \frac{1}{1000}, 50\right\}, \text{PlotStyle} \to \text{Hue}[0]\};$$

$$\left\{p''[h] == -\frac{4\,M^2\,\pi\,\gamma\,p[h]^2}{R^2\,T^2} - \frac{2\,p'[h]}{h} + \frac{p'[h]^2}{p[h]}\right\}$$



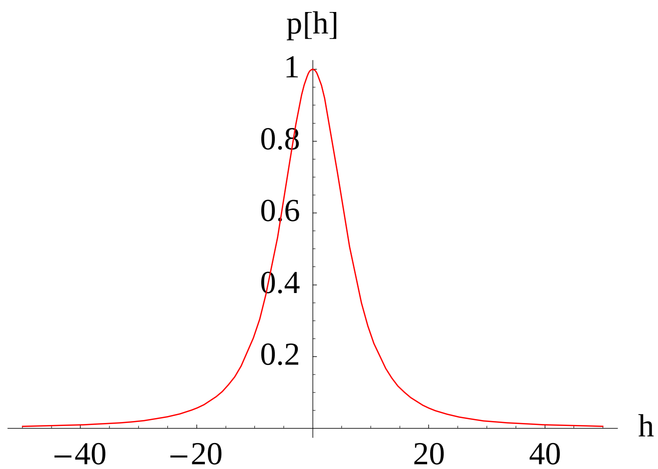
```
Solve[PressureEquation[p, 1], p''[h]] /. {Rule → Equal} // Flatten // Simplify
Curve[LeftHandSide, 2] =
   NDSolve[{%[[1]] /. {M → 1, γ → 1, R → 1, T → 10}, p[-1/1000] == 1, p'[-1/1000] == 0},
     p[h], {h, -1/1000, -50}];
Figure[LeftHandSide, 2] = Plot @@ {Evaluate[p[h] /. Curve[LeftHandSide, 2]],
       {h, -1/1000, -50}, PlotStyle → Hue[0]};
```

$$\left\{p''[h] == -\frac{4 M^2 \pi \gamma\, p[h]^2}{R^2 T^2} - \frac{2 p'[h]}{h} + \frac{p'[h]^2}{p[h]}\right\}$$



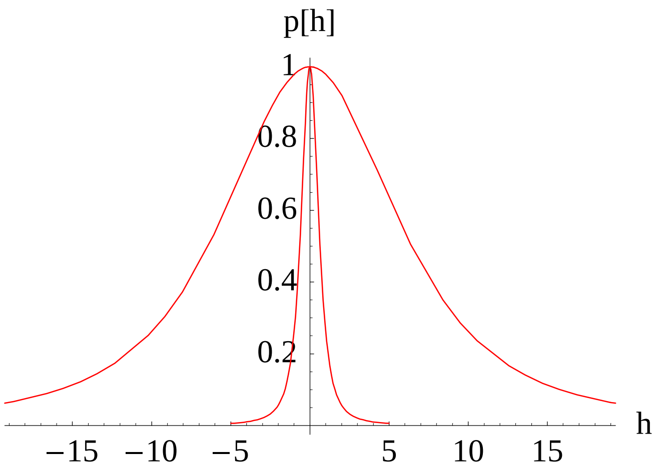
The superposition of both of these curves also is very convincing and thoroughly similar:

```
Figure[2] = Show[Figure[LeftHandSide, 2], Figure[RightHandSide, 2], AxesLabel → {"h", "p[h]"}];
```



The similarity of the figures can be verified:

Show[Figure[1], Figure[2]];



By this the fundamental shape of the pressure distriution of a star is described, if the same consists of one gas only at only one temperature.

At the border to the vacuum concerning temperature transport a total reflexion takes place, thus also small gas amounts at low temperature can consist thoroughly stable in space as a "gas drop". A critical value, since which a star can consist, has not been found.

A gas cloud in space looses its temperature by radiation only.

The asymptotics $p[\pm\infty] \to 0$ affirm that a gas in principle fills the whole space that is available.

■ 1.3.3.7. Erreaneous Simplification

The following analytical simplification of the equation has been tried:

$$\frac{\#}{h\, p[h]} \;\&\; /@\; \text{PressureEquation}[p, 1] \;//\; \text{ExpandAll}$$

$$\text{PressureEquation}[p, 2] = \%\; /.\; \{p \to \text{Function} @@ \{\{h\}, e^{f[h]}\}\} \;//\; \text{ExpandAll}$$

$$\frac{2\, p'[h]}{h\, p[h]} - \frac{p'[h]^2}{p[h]^2} + \frac{p''[h]}{p[h]} == -\frac{4\, M^2 \pi \gamma\, p[h]}{R^2\, T^2}$$

$$\frac{2\, f'[h]}{h} + f''[h] == -\frac{4\, E^{f[h]}\, M^2 \pi \gamma}{R^2\, T^2}$$

This is a non-linear differential equation of 2nd order in $f[h]$ that cannot be solved easily.

However, this equation type leads to a solution that could be $f[h] \to \text{Log}[p[h]]$, e.g.:

$$\text{PressureEquation}[p, 2]\; /.\; \left\{f \to \text{Function}\left[\{h\}, \text{Log}\left[\frac{C[1]}{h^2 + C[2]}\right]\right]\right\} \;//\; \text{ExpandAll} \;//\; \text{Simplify}$$

$$-\frac{2\,(h^2 + 3\, C[2])}{(h^2 + C[2])^2} == -\frac{4\, M^2 \pi \gamma\, C[1]}{R^2\, T^2\, (h^2 + C[2])}$$

■ 1.3.3.8. Differential Equation of the Reciprocal Function

The differential equation of the reciprocal function seems to be easier:

PressureEquation[*p*, 1] /. {*p* → Function[{*h*}, $\frac{1}{g[h]}$]} // ExpandAll

PressureEquation[*p*, 3] = $\frac{\# g[h]^2}{h}$ & /@ % // ExpandAll

$$-\frac{2 g'[h]}{g[h]^2}+\frac{h g'[h]^2}{g[h]^3}-\frac{h g''[h]}{g[h]^2}==-\frac{4 h M^2 \pi \gamma}{R^2 T^2 g[h]^2}$$

$$-\frac{2 g'[h]}{h}+\frac{g'[h]^2}{g[h]}-g''[h]==-\frac{4 M^2 \pi \gamma}{R^2 T^2}$$

Here a trial by a polynomial is motivated:

PressureEquation[*p*, 3] /. {*g* → Function[{*h*}, *a h*²]}
Solve[%, *a*] // Flatten
PressureSolution[*p*, 3, 1] = {*g* → Function @@ {{*h*}, *a h*² /. %}}
PressureEquation[*p*, 3] /. %

$$-2 a==-\frac{4 M^2 \pi \gamma}{R^2 T^2}$$

$$\left\{a \to \frac{2 M^2 \pi \gamma}{R^2 T^2}\right\}$$

$$\left\{g \to \text{Function}\left[\{h\}, \frac{2 h^2 M^2 \pi \gamma}{R^2 T^2}\right]\right\}$$

True

This function owns a horizontal tangent **$g'[0] \to 0$**.

By this follows for the pressure:

PressureSolution[*p*, 1, 1] = {*p* → Function @@ {{*h*}, $\frac{1}{g[h]}$ /. PressureSolution[*p*, 3, 1]}}

PressureEquation[*p*, 1] /. %

$$\left\{p \to \text{Function}\left[\{h\}, \frac{R^2 T^2}{2 h^2 M^2 \pi \gamma}\right]\right\}$$

True

Thus a *hydrostatic stable state* with **$p[0] \to \infty$** and **$p'[0] \to \infty$** exists. In the immediate neighborhood of this double singularity everything is regular.

■ 1.3.3.9. Interpretation Problems

For the representatives of the *Big Bang* now is problematic, that a *hydrostatic stabile state* has been calculated.

Now for an anyhow compressible gas results, that the same in the center is compressed due to the ideal gas law to an infinitly high pressure. Of course, this result questiones the limits of the ideal gas law and gives the hint, that at least the Van der Waals equation is needed to be precise.

However the following dependency of the fall acceleration $g[h]$ is found:

$$g[h] == -\frac{\gamma\, 4\,\pi}{h^2}\int_0^h \rho[h]\, h^2\, dh \;/.\; \left\{\rho[h] \to \frac{M\; p[h]}{R\,T}\right\}$$

% /. PressureSolution[*p*, 1, 1]

$$g[h] == -\frac{4\,M\,\pi\,\gamma \int_0^h h^2\; p[h]\, dh}{h^2\,R\,T}$$

$$g[h] == -\frac{2\,R\,T}{h\,M}$$

This result tells, that a *Black Hole* exists especially due to the ideal gas law and *does not suck anything into itself* that has come too near, because a *hydrostatic stabile state* has been calculated.

The mass of such a *Black Hole* yields:

$$m[h] == 4\,\pi \int_0^h \rho[h]\, h^2\, dh \;/.\; \left\{\rho[h] \to \frac{M\; p[h]}{R\,T}\right\}$$

% /. PressureSolution[*p*, 1, 1]

$$m[h] == \frac{4\,M\,\pi \int_0^h h^2\; p[h]\, dh}{R\,T}$$

$$m[h] == \frac{2\,h\,R\,T}{M\,\gamma}$$

This means, that for $h \to 0$ also the concerning mass $m[h]$ disappeares, because nothing also weighs nothing.

Indeed, here the problem is an invalid use of the ideal gas law: Each gas consists of as incompressible assumed particles, of which the total volume cannot fall short of by compression.

Therefore the theory of the *Big Bang* and the *Black Holes* cannot be verified by the here given calculation.

■ 1.3.3.10. Van der Waals Law

The van der Waals law is ([dtv1969], volume **10**, keyword *Zustandsgleichungen d) van-der-Waalsche Z. [condition equations, d) van der Waals condition equation]*, page 198-199):

$$\left(p + \frac{a}{V^2}\right)(V - b) == n\,R\,T\ /.\ \left\{n \to \frac{m}{M}\right\}\ /.\ \{m \to \rho\,V\}$$

Solve[%, ρ] // Flatten

% //. UnitCheck

$$\left(p + \frac{a}{V^2}\right)(-b + V) == \frac{R\,T\,V\,\rho}{M}$$

$$\left\{\rho \to \frac{M\,(-b + V)\,(a + p\,V^2)}{R\,T\,V^3}\right\}$$

{True}

Here, the volume of a sphere is given strictly geometric by

$$\text{SphereVolume}[h] = \left\{V \to 4\,\pi \int_0^h h^2\, dh\right\}$$

$$\left\{V \to \frac{4\,h^3\,\pi}{3}\right\}$$

Thus $\rho[h]$ can be given also due to the van der Waal law as a function of h and $p[h]$:

$$\text{VanDerWaalsDensity}[h] = \left\{\rho[h] \to \frac{M\,(-b + V)\,(a + p[h]\,V^2)}{R\,T\,V^3}\right\}\ /.\ \text{SphereVolume}[h]$$

% //. UnitCheck

$$\left\{\rho[h] \to \frac{27\,M\left(-b + \frac{4\,h^3\,\pi}{3}\right)\left(a + \frac{16}{9}\,h^6\,\pi^2\,p[h]\right)}{64\,h^9\,\pi^3\,R\,T}\right\}$$

{True}

By this follows in the next step the fall accelerationg $g[h]$ and the corresponding integrodifferential equation of $p[h]$:

$$\left\{g[h] \to -\frac{\gamma\,4\,\pi}{h^2}\int_0^h \rho[h]\,h^2\, dh\right\}$$

% /. VanDerWaalsDensity[h]

PressureEquation[p, 4] = p'[h] == g[h] ρ[h] /. % /. VanDerWaalsDensity[h]

$$\left\{g[h] \to -\frac{4\,\pi\,\gamma \int_0^h h^2\,\rho[h]\, dh}{h^2}\right\}$$

$$\left\{g[h] \to -\frac{27\,M\,\gamma \int_0^h \frac{\left(-b + \frac{4\,h^3\,\pi}{3}\right)\left(a + \frac{16}{9}\,h^6\,\pi^2\,p[h]\right)}{h^7}\, dh}{16\,h^2\,\pi^2\,R\,T}\right\}$$

$$p'[h] == -\frac{729\,M^2\left(-b + \frac{4\,h^3\,\pi}{3}\right)\gamma\left(\int_0^h \frac{\left(-b + \frac{4\,h^3\,\pi}{3}\right)\left(a + \frac{16}{9}\,h^6\,\pi^2\,p[h]\right)}{h^7}\, dh\right)\left(a + \frac{16}{9}\,h^6\,\pi^2\,p[h]\right)}{1024\,h^{11}\,\pi^5\,R^2\,T^2}$$

After isolating of the integral a derivative can be applied:

$$\frac{\# h^{11}}{\left(a+\frac{16}{9}h^6\pi^2 p[h]\right)\left(-b+\frac{4h^3\pi}{3}\right)} \text{ \& /@ PressureEquation}[p, 4]$$

∂_h **# & /@ % // Simplify**

$$\text{PressureEquation}[p, 5] = \frac{\#}{27 h^{10}}\left((3b-4h^3\pi)^2 (9a+16h^6\pi^2 p[h])^2\right) \text{ \& /@ \% // ExpandAll}$$

$$\frac{h^{11}\, p'[h]}{\left(-b+\frac{4h^3\pi}{3}\right)\left(a+\frac{16}{9}h^6\pi^2 p[h]\right)} == -\frac{729 M^2\gamma \int_0^h \frac{\left(-b+\frac{4h^3\pi}{3}\right)\left(a+\frac{16}{9}h^6\pi^2 p[h]\right)}{h^7}\, dh}{1024\pi^5 R^2 T^2}$$

$$\Big(27 h^{10}\big((-297ab+288ah^3\pi+16h^6\pi^2(-15b+8h^3\pi)p[h])p'[h]+16h^7\pi^2(3b-4h^3\pi)p'[h]^2+h(-3b+4h^3\pi)(9a+16h^6\pi^2 p[h])p''[h]\big)\Big)\Big/\left((3b-4h^3\pi)^2(9a+16h^6\pi^2p[h])^2\right) == \frac{27M^2(3b-4h^3\pi)\gamma(9a+16h^6\pi^2p[h])}{1024h^7\pi^5R^2T^2}$$

$$-297ab\,p'[h]+288ah^3\pi\,p'[h]-240bh^6\pi^2 p[h]\,p'[h]+128h^9\pi^3 p[h]\,p'[h]+48bh^7\pi^2 p'[h]^2-64h^{10}\pi^3 p'[h]^2-27abh\,p''[h]+36ah^4\pi\,p''[h]-48bh^7\pi^2 p[h]\,p''[h]+64h^{10}\pi^3 p[h]\,p''[h] ==$$
$$\frac{19683a^3b^3M^2\gamma}{1024h^{17}\pi^5R^2T^2}-\frac{19683a^3b^2M^2\gamma}{256h^{14}\pi^4R^2T^2}+\frac{6561a^3bM^2\gamma}{64h^{11}\pi^3R^2T^2}-\frac{729a^3M^2\gamma}{16h^8\pi^2R^2T^2}-\frac{243a^2M^2\gamma\,p[h]}{h^2R^2T^2}+$$
$$\frac{6561a^2b^3M^2\gamma\,p[h]}{64h^{11}\pi^3R^2T^2}-\frac{6561a^2b^2M^2\gamma\,p[h]}{16h^8\pi^2R^2T^2}+\frac{2187a^2bM^2\gamma\,p[h]}{4h^5\pi R^2T^2}-\frac{729ab^2M^2\gamma\,p[h]^2}{h^2R^2T^2}+$$
$$\frac{729ab^3M^2\gamma\,p[h]^2}{4h^5\pi R^2T^2}+\frac{972abhM^2\pi\gamma\,p[h]^2}{R^2T^2}-\frac{432ah^4M^2\pi^2\gamma\,p[h]^2}{R^2T^2}+\frac{108b^3hM^2\pi\gamma\,p[h]^3}{R^2T^2}-$$
$$\frac{432b^2h^4M^2\pi^2\gamma\,p[h]^3}{R^2T^2}+\frac{576bh^7M^2\pi^3\gamma\,p[h]^3}{R^2T^2}-\frac{256h^{10}M^2\pi^4\gamma\,p[h]^3}{R^2T^2}$$

This is the real equation, the solution of which estimates the actual pressure distribution in a star better than the theory of the *Big Bang* or of the *Black Holes*.

For $a \to 0$ the differential equation is simplified:

$$\text{PressureEquation}[p, 6] = \frac{\#}{\pi^2\, 16 h^6} \text{ \& /@ PressureEquation}[p, 5] \text{ /. } \{a \to 0\} \text{ // Simplify}$$

$$h(3b-4h^3\pi)\,p'[h]^2 + p[h]\left((-15b+8h^3\pi)\,p'[h]+h(-3b+4h^3\pi)\,p''[h]\right) == \frac{M^2(3b-4h^3\pi)^3\gamma\,p[h]^3}{4h^5\pi R^2T^2}$$

This equation describes a gas clearly above the critical point, i.e. at high enough temperature.

Also this equation is of low enjoyability and leaves yet new ground for enough research generations.

■ 1.3.4. Discussion

The presented calculation shows, that gas volumina of any scale alltogether can exist stable in space and nevertheless always loose some gas to an empty space.

The here discussed considerations are valid for a constant temperature T, where a cooling into space mainly happens by radiation, because at the bordering line to the vacuum a total reflexion of temperature transport takes place.

Neither the idea of a *Big Bang*, nor the idea of *Black Holes* that suck everything into themselves agree to here even for the ideal gas law found solution of a stable pressure distribution in a star, or in a gas cloud respectively.

There is gain to calculate a check to a singular result and to use the full interpretation scope.

■ 1.4. Protocol

The *Mathematica* version was:

{$Version, $ReleaseNumber, $LicenseID}

{Microsoft Windows 3.0 (October 6, 1996), 0, L4526−3546}

The calculation time was:

TimeUsed[ ] "s"

52.26 s

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